# Restoring time-reversal covariance in relaxed hydrodynamics

Andrea Amoretti [1,2,*] Daniel K. Brattan [1,3,†] Luca Martinoia [1,2,‡] and Ioannis Matthaiakakis [1,2,§]

[1] *Dipartimento di Fisica, Università di Genova, via Dodecaneso 33, I-16146, Genova, Italy*
[2] *I.N.F.N. - Sezione di Genova, via Dodecaneso 33, I-16146, Genova, Italy*
[3] *CPHT, CNRS, École polytechnique, Institut Polytechnique de Paris, 91120 Palaiseau, France*

(Received 15 May 2023; accepted 18 August 2023; published 1 September 2023)

In hydrodynamics, for generic relaxations, the stress tensor and $U(1)$ charge current two-point functions are not time-reversal covariant. This remains true even if the Martin-Kadanoff procedure happens to yield Onsager reciprocal correlators. We consider linearized relativistic hydrodynamics on Minkowski space in the presence of energy, $U(1)$ charge, and momentum relaxation. We then show how one can find the minimal relaxed hydrodynamic framework that does yield two-point functions consistent with time-reversal covariance. We claim the same approach naturally applies to boost agnostic hydrodynamics and its limits (e.g., Carrollian, Galilean, and Lifshitz fluids).

DOI: 10.1103/PhysRevD.108.056003

## I. INTRODUCTION

When we couple a charged fluid to an external electric field in order to compute the fluid's optical conductivities, we find that these conductivities diverge in the strict direct current (DC) limit $\omega \to 0$ [1]. The origin of this divergence traces back to the existence of a conserved momentum operator, which implies that a constant electric field (or temperature gradient) adds momentum to the fluid without bound. The standard approach to solve this problem is the introduction of a small relaxation parameter $\tau_p^{-1}$ which breaks translation invariance and relaxes the total momentum. While momentum is no longer conserved, it can still be a relevant hydrodynamic charge if $\tau_p$ is large enough [1] (for some holographic examples see [2–6]).

In the presence of a $U(1)$ axial anomaly, momentum relaxation is not enough to obtain finite DC conductivities. In this case one must also relax energy and $U(1)$ charge conservation [7–10]. The inclusion of these relaxation terms is natural, in the sense that energy relaxation is a fact of certain condensed matter processes and charge relaxation may appear when the corresponding symmetry is only approximate, see, e.g., [11–14] for a kinetic theory derivation and [15] for a discussion of relaxed nondissipative hydrodynamics.

At the computational level, there are two main approaches to obtaining the response functions and their corresponding conductivities in linearized hydrodynamics: the variational or background field approach [16] and the Martin-Kadanoff or canonical approach [17]. In the first the fluid is perturbed by spacetime and gauge field fluctuations that couple to the full stress-energy tensor and current, respectively, while in the second sources only for the conserved charge densities are introduced.

Because the relaxation terms explicitly break the Lorentz symmetry of the system, the variational approach is usually deemed impractical for computing the Green's functions [18] because, in general, it is impossible to write the relaxation terms in a *unique* covariant form without introducing additional fields, which one may or may not assume to be dynamical. Examples of such fields include a massive Goldstone field or a background vector (see, e.g., [19,20]). On the other hand, the Martin-Kadanoff method works well in the presence of relaxation terms, but comes short on other aspects. In particular, (i) it does not give access to all the Green functions, but only to those related to the thermodynamic charges [21] and (ii) in more general cases, e.g., when there are strong electric and magnetic fields that polarize the fluid, it might not be straightforward to obtain the relations between the charges and their thermodynamic sources. For these reasons, it is of interest to find a unique prescription to study the response functions of fluids with weakly relaxed charges via the variational method.

In this paper we present a prescription that allows one to compute the two-point functions of relaxed hydrodynamics via the background field method. Namely, we consider a

*andrea.amoretti@ge.infn.it
†danny.brattan@gmail.com
‡luca.martinoia@ge.infn.it
§ioannis.matthaiakakis@edu.unige.it

particular example, relaxed relativistic hydrodynamics with a $U(1)$ charge and introduce all the possible source terms (metric and gauge field fluctuations) to the conservation equations at order one in fluctuations and order zero in derivatives. We find that by simply requiring time-reversal covariance of the microscopic theory we can completely constrain all the extra parameters we introduce. The relaxation terms that survive this procedure exactly reproduce the Green's functions obtained via the Martin-Kadanoff approach up to contact terms.

The paper is structured as follows: in Sec. II we discuss the general properties of constant relaxations in the Martin-Kadanoff formalism for an example $U(1)$ charged relativistic fluid and obtain a set of very general constraints based on Onsager reciprocity, positivity of entropy production, and linearized stability. In Sec. III we implement the same relaxations in the variational approach and, by imposing time-reversal covariance on the correlators, show how to obtain consistent results between the prescriptions.

## II. THE MARTIN-KADANOFF APPROACH AND ONSAGER RECIPROCITY

In the present section, we consider the Martin-Kadanoff procedure in the presence of the most general sources of relaxation linear in the fluctuation fields. This approach gives a subset of the possible two-point functions of the stress tensor and $U(1)$ charge current. Subsequently, we impose Onsager reciprocity on the obtained Green's functions alongside the second law of thermodynamics. This naturally leads to strong constraints on the allowed relaxation terms. Having developed this framework, in the next section we shall explore how to make this effective description time-reversal covariant for all the two-point functions of the currents one can calculate in hydrodynamics.

Our starting point is the linearized hydrodynamic equations of motion, Fourier transformed in the spatial directions. They can generically be brought into the following form:

$$\partial_t\varphi_a(t,\mathbf{k}) + M_{ab}(\mathbf{k})\varphi_b(t,\mathbf{k}) = 0. \tag{1}$$

By $\varphi_a$ we denote the fluctuations of the conserved charges, while $M_{ab}$ is the hydrodynamic matrix, whose specific expression depends on the constitutive relations and the equations of motion. Given $M_{ab}$ we can write down an explicit formula for the retarded Green's function $G^R_{ab}$, namely [16]

$$G^R_{ab}(z,\mathbf{k}) = -(1 + izK^{-1}(z,\mathbf{k}))_{ac}\chi_{cb}, \tag{2}$$

where we defined $K_{ab} = -iz\delta_{ab} + M_{ab}(\mathbf{k})$, and $\chi_{ab}$ is the thermodynamic susceptibility matrix,

$$\chi_{ab} = \frac{\partial\varphi_a}{\partial\lambda_b}. \tag{3}$$

We denote by $\lambda_a$ the sources conjugate to the charges.

In addition to any continuous symmetries, discrete symmetries (such as time-reversal covariance) should be imposed on the hydrodynamic effective theory if they are present in the microscopic theory. Such discrete symmetries often lead to constraints on transport coefficients. In relaxed hydrodynamics appropriate relaxations are chosen to break the relevant continuous symmetries, but what is not immediately obvious is whether they break the discrete symmetries—after all, the discrete symmetries are not part of the proper Lorentz group. Following [16], we will assume that the relaxed microscopic theory does preserve microscopic time-reversal covariance and thus require

$$G^R_{ab}(\omega,\mathbf{k}) = \eta_a\eta_b G^R_{ba}(\omega,-\mathbf{k}) \tag{4}$$

of our hydrodynamic correlations functions, where $\eta_a$ is the time-reversal eigenvalue [22] for the field $\varphi_a$. Enforcing time-reversal using (2), leads to the constraint [23]

$$\chi S M^T(-\mathbf{k}) = M(\mathbf{k})\chi S, \tag{5}$$

with $S = \mathrm{diag}(\eta_1, \eta_2, \ldots)$ the matrix of time-reversal eigenvalues of the $\varphi$s. In writing (5), we have assumed that there are no explicit parameters $B$, such as the magnetic field, that break time-reversal invariance of the microscopic theory. In such cases one may be able to extend (5) to a relationship between different theories where the parameter $B$ also transforms appropriately under time reversal. For example, in the case of a constant magnetic field we take $B \to -B$ under time reversal.

The above applies for general hydrodynamic theories; now we specialize our discussion to relativistic hydrodynamics. We consider a charged relativistic fluid at temperature $T$ and chemical potential $\mu$ propagating in Minkowski spacetime. Its energy-momentum tensor $T^{\mu\nu}$ and electric current $J^\mu$ are given in the Landau frame by [16]

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + P\Delta^{\mu\nu} - \eta\Delta^{\mu\alpha}\Delta^{\nu\beta}\left(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{d}\eta_{\alpha\beta}\partial_\lambda u^\lambda\right) - \zeta\partial_\mu u^\mu + \mathcal{O}(\partial^2), \tag{6a}$$

$$J^\mu = nu^\mu - \sigma\Delta^{\mu\nu}T\partial_\nu\frac{\mu}{T} + \mathcal{O}(\partial^2), \tag{6b}$$

with $\Delta^{\mu\nu} = \eta^{\mu\nu} + u^\mu u^\nu$, the projector normal to the velocity profile $u^\mu$, $\eta^{\mu\nu} = \mathrm{diag}(-1, 1, 1\ldots 1)$, and $d$ the number of spatial dimensions.

The hydrodynamic equations corresponding to $T^{\mu\nu}$, $J^\mu$ are the energy-momentum and charge conservation equations, $\partial_\mu T^{\mu\nu} = 0 = \partial_\mu J^\mu$. We linearize these equations around a state with constant energy density, number density, and zero spatial velocity,

$$u^\mu = u^\mu_{(0)} + \delta u^\mu, \qquad \epsilon = \epsilon_{(0)} + \delta\epsilon, \qquad n = n_{(0)} + \delta n, \tag{7}$$

with $u^\mu_{(0)} = (-1, 0, \ldots, 0)$ in a Cartesian coordinate system. Subsequently, we add relaxation terms [24] that are linear in the charge fluctuations to find

$$\partial_t \delta\epsilon + (\epsilon_{(0)} + P_{(0)})\partial_i \delta v^i = -\left(\frac{1}{\tau_{\epsilon\epsilon}}\delta\epsilon + \frac{1}{\tau_{\epsilon n}}\delta n\right), \tag{8a}$$

$$\partial_t \delta n + n_{(0)}\partial_i \delta v^i - \sigma T_{(0)}\partial_i^2\left(\delta\frac{\mu}{T}\right) = -\left(\frac{1}{\tau_{n\epsilon}}\delta\epsilon + \frac{1}{\tau_{nn}}\delta n\right), \tag{8b}$$

$$\partial_t \delta p^i + \partial^i \delta P - \eta\left(\partial_j^2 \delta v^i + \frac{1}{3}\partial^i\partial_j \delta v^j\right) - \zeta\partial_i\partial_j \delta v^j = -\frac{1}{\tau_p}\delta p^i, \tag{8c}$$

with $\delta p_i = (\epsilon_{(0)} + P_{(0)})\delta v_i$. These equations (8) correspond to the (non)conservation of energy, charge, and momentum, respectively. The energy and charge relaxation terms are defined intuitively in terms of their corresponding relaxation times. In particular, $\tau_p$ is the momentum relaxation time, while $\tau_{\epsilon\epsilon}$ and $\tau_{nn}$ are the energy and charge relaxations times. The terms $\tau_{n\epsilon}$, $\tau_{\epsilon n}$ are mixed relaxations, which as written do not have the units of time, and one can find an example of such terms in [9]. We could also consider generalizing momentum relaxation to mix different momentum components, however, this type of relaxation can always be reduced to the one found in (8) (see Sec. 2.4 of [15]). Thus, we have added to the right-hand sides of (8), all possible relaxation terms linear in fluctuations and without explicit derivatives. Subsequently, we take the relaxations times to be small ($\tau$ large) by which we mean, in the linearized regime, that $\tau^{-1} \sim \mathcal{O}(\partial)$. Finally, we note that the displayed expressions (8) are agnostic on the specific relaxation scenario, therefore, they are quite general and can capture many different situations.

Before we discuss in detail the constraints on relaxation stemming from time reversal and the second law, we note that a heuristic thermodynamic argument already suggests the existence of such constraints. Following [7], consider a scattering event between the fluid microscopic constituents and some impurities/defects that relax the energy (or charge). Because of the scatterings, the fluid loses $\delta\epsilon$ energy in a time $\tau_{\epsilon\epsilon}$. However, this implies that it also loses charge $\delta n = \frac{\partial n}{\partial \epsilon}\delta\epsilon = \frac{\delta\epsilon}{\mu_{(0)}}$ in the same time interval. An analogous argument tells us that if the fluid loses $\delta n$ charge, then it also loses $\delta\epsilon = \mu_{(0)}\delta n$ energy. Hence we see that, in general, energy and charge relaxations are strongly intertwined, and we can already put forward the ansatz

$$\frac{\mu_{(0)}}{\tau_{nn}} = \frac{1}{\tau_{\epsilon n}}, \qquad \frac{1}{\tau_{\epsilon\epsilon}} = \frac{\mu_{(0)}}{\tau_{n\epsilon}}. \tag{9}$$

We confirm this holds generically for relaxed theories that preserve positivity of entropy production shortly.

We impose the Onsager relations to hydrodynamics by requiring (5) on the equations of motion (8). Taking $\varphi_a(t, \mathbf{k}) = (\delta\epsilon, \delta n, \delta p_i)$ and considering, without loss of generality, $\mathbf{k} = (k_x, 0, 0)$ we determine the matrix $M_{ab}$ of (1) to be

$$M = \begin{pmatrix} \frac{1}{\tau_{\epsilon\epsilon}} & \frac{1}{\tau_{\epsilon n}} & ik_x & 0 & 0 \\ k_x^2\sigma\beta_\epsilon + \frac{1}{\tau_{n\epsilon}} & k_x^2\sigma\beta_n + \frac{1}{\tau_{nn}} & \frac{ik_x n_{(0)}}{P_{(0)}+\epsilon_{(0)}} & 0 & 0 \\ ik_x\frac{\partial P}{\partial\epsilon} & ik_x\frac{\partial P}{\partial n} & \frac{k_x^2(3\zeta+4\eta)}{3(P_{(0)}+\epsilon_{(0)})} + \frac{1}{\tau_p} & 0 & 0 \\ 0 & 0 & 0 & \frac{k_x^2\eta}{P_{(0)}+\epsilon_{(0)}} + \frac{1}{\tau_m} & 0 \\ 0 & 0 & 0 & 0 & \frac{k_x^2\eta}{P_{(0)}+\epsilon_{(0)}} + \frac{1}{\tau_m} \end{pmatrix}, \tag{10}$$

where we defined $\beta_\epsilon = \frac{\partial\mu}{\partial\epsilon} - \frac{\mu_{(0)}}{T_{(0)}}\frac{\partial T}{\partial\epsilon}$, $\beta_n = \frac{\partial\mu}{\partial n} - \frac{\mu_{(0)}}{T_{(0)}}\frac{\partial T}{\partial n}$, and the matrix of the time-reversal eigenvalue is $S = \mathrm{diag}(1, 1, -1, -1, -1)$. We find

$$\frac{\chi_{\epsilon\epsilon}}{\tau_{n\epsilon}} - \frac{\chi_{\epsilon n}}{\tau_{\epsilon\epsilon}} + \frac{\chi_{n\epsilon}}{\tau_{nn}} - \frac{\chi_{nn}}{\tau_{\epsilon n}} = 0, \tag{11}$$

where the susceptibilities are

$$\chi_{n\epsilon} = \chi_{\epsilon n} = \frac{\partial\epsilon}{\partial\mu} = T_{(0)}\frac{\partial n}{\partial T} + \mu_{(0)}\frac{\partial n}{\partial\mu}, \qquad \chi_{nn} = \frac{\partial n}{\partial\mu},$$
$$\chi_{\epsilon\epsilon} = T_{(0)}\frac{\partial\epsilon}{\partial T} + \mu_{(0)}\frac{\partial\epsilon}{\partial\mu}. \tag{12}$$

The thermodynamic derivatives are taken in the grand canonical ensemble, at fixed $\mu$ or $T$, respectively. We can see then that if we set $\tau_{n\epsilon}^{-1} = \tau_{\epsilon n}^{-1} = 0$ leaving only the pure charge and energy relaxations typically encountered in the literature, then we must impose $\tau_{\epsilon\epsilon} = \tau_{nn}$. This relation is known [8], but we shall now see that this truncated relaxation is at odds with the second law of thermodynamics.

The linearized entropy current is given by

$$\delta s^\mu = \delta s^\mu_{\mathrm{can}} + \delta s^\mu_{\mathrm{eq}}, \tag{13a}$$

$$s^\mu_{\mathrm{can}} = \frac{1}{T}(P u^\mu + T^{\mu\nu}u_\nu - \mu J^\mu), \tag{13b}$$

where $s^\mu_{\mathrm{eq}}$ is present to ensure that the hydrodynamic equations vanish upon imposition of hydrostaticity and is zero at the relevant order in our current situation. It is then not difficult to show that the divergence of this current takes the form

$$T_{(0)}\partial_\mu\delta s^\mu = \delta\epsilon\left(\frac{\mu_{(0)}}{\tau_{n\epsilon}} - \frac{1}{\tau_{\epsilon\epsilon}}\right) + \delta n\left(\frac{\mu_{(0)}}{\tau_{nn}} - \frac{1}{\tau_{\epsilon n}}\right) + \mathcal{O}(\delta^2, \partial^2). \tag{14}$$

Positivity of entropy production requires that the divergence of the entropy current be positive definite on any (including the linearized) solution of the hydrodynamic equations of motion. As each of the fluctuations $\delta\epsilon$ and $\delta n$ in (14) are of arbitrary sign, it follows that their coefficients must be zero [25]. This gives two constraints on the relaxation rates, and hence we obtain (9), confirming that the relaxations of energy and charge are connected for thermodynamic reasons. Subsequently, with these constraints between the relaxation rates due to the second law, we can use (11) to find

$$\frac{\partial\epsilon}{\partial T}\frac{1}{\tau_{\epsilon\epsilon}} + \frac{\partial n}{\partial T}\frac{\mu_{(0)}}{\tau_{nn}} = 0. \tag{15}$$

This leaves a one parameter family of relaxations, which we can parametrize by $\tau_{nn}$. Notice that, in general, and this will be confirmed by the study of the modes, not all relaxation times must be positive.

Equations (14) and (15) fix all but one of the relaxation terms giving us a one-parameter family that, at least at the linearized level, satisfy positivity of entropy production and Onsager reciprocity. Importantly, we can see that, unless the chemical potential is zero, we will find $\tau_{\epsilon n} \neq 0$ whenever $\tau_{nn} \neq 0$, if we want these properties to hold. Alternatively, one can have more generic relaxation rates if the second law is ignored. In this case, the relaxation rates represent the coupling of a fluid to an open system, rather than some UV degrees of freedom in a more complete quantum theory with a quasihydrodynamic limit (see, e.g., the discussion in [26–29]).

In (14) we examined the entropy positivity condition at lowest order in fluctuations. However, when considering linearized hydrodynamics, constraints on transport coefficients can be inferred only by examining the second law at order two in fluctuations. In our case, at order two in fluctuations, the divergence of the entropy current is given by

$$T_{(0)}\partial_\mu\delta s^\mu = \frac{\delta\epsilon\delta T}{T_{(0)}\tau_{\epsilon\epsilon}} + \frac{\delta n\delta T}{T_{(0)}\tau_{\epsilon n}} + \frac{\delta\mu\delta n}{\tau_{nn}} + \frac{\delta\mu\delta\epsilon}{\tau_{n\epsilon}} - \mu_{(0)}\frac{\delta T\delta n}{T_{(0)}\tau_{nn}} - \mu_{(0)}\frac{\delta T\delta\epsilon}{T_{(0)}\tau_{n\epsilon}} + \frac{1}{\tau_p}(\epsilon_{(0)} + P_{(0)})\delta v^2 + \sigma\left(\frac{\mu_{(0)}}{T_{(0)}}\partial\delta T - \partial\delta\mu\right)^2$$
$$+ \eta(\delta\sigma^{ij})^2 + \zeta(\partial_i\delta v^i)^2 + \mathcal{O}(\delta^3, \partial^3). \tag{16}$$

Positivity of the rhs gives the usual constraints on the transport coefficients $\sigma \geq 0$, $\eta \geq 0$, and $\zeta \geq 0$. Furthermore, we also find $\tau_p \geq 0$ as expected for momentum relaxation. We can rewrite the remaining relaxations in terms of fluctuations of just $T$ and $\mu$ to find

$$T_{(0)}\partial_\mu\delta s^\mu = \delta\mu\delta T\left(\frac{\partial\epsilon}{\partial\mu}\frac{1}{T_{(0)}\tau_{\epsilon\epsilon}} + \frac{\partial n}{\partial\mu}\frac{1}{T_{(0)}\tau_{\epsilon n}} + \frac{\partial n}{\partial T}\frac{1}{\tau_{nn}} + \frac{\partial\epsilon}{\partial T}\frac{1}{\tau_{n\epsilon}} - \frac{\mu_{(0)}}{T_{(0)}}\frac{\partial n}{\partial\mu}\frac{1}{\tau_{nn}} - \frac{\mu_{(0)}}{T_{(0)}}\frac{\partial\epsilon}{\partial\mu}\frac{1}{\tau_{n\epsilon}}\right)$$
$$+ \frac{\delta T^2}{T_{(0)}}\left(\frac{\partial\epsilon}{\partial T}\frac{1}{\tau_{\epsilon\epsilon}} + \frac{\partial n}{\partial T}\frac{1}{\tau_{\epsilon n}} - \frac{\partial n}{\partial T}\frac{\mu_{(0)}}{\tau_{nn}} - \frac{\partial\epsilon}{\partial T}\frac{\mu_{(0)}}{\tau_{n\epsilon}}\right) + \delta\mu^2\left(\frac{\partial n}{\partial\mu}\frac{1}{\tau_{nn}} + \frac{\partial\epsilon}{\partial\mu}\frac{1}{\tau_{n\epsilon}}\right) + \ldots, \tag{17}$$

where the dots are the standard dissipative terms. Using the constraints (9), and the results of (15), we can simplify the above expression (17) to

$$T_{(0)}\partial_\mu \delta s^\mu = \delta\mu^2\left(\frac{\partial n}{\partial\mu}\frac{1}{\tau_{nn}} + \frac{\partial\epsilon}{\partial\mu}\frac{1}{\tau_{n\epsilon}}\right) + \dots. \quad (18)$$

The remnant nonzero term can be written in terms of the susceptibilities as

$$\frac{\delta\mu^2}{\tau_{nn}}(\chi_{\epsilon\epsilon}\chi_{nn} - \chi_{\epsilon n}^2) \lesseqgtr 0, \quad (19)$$

where the sign of the inequality depends on the sign of $\frac{\partial\epsilon}{\partial T}$. Because the susceptibility matrix is positive definite, the bracket is also positive. Hence when $\frac{\partial\epsilon}{\partial T} \geq 0$ then $\tau_{nn} \geq 0$, while if $\frac{\partial\epsilon}{\partial T} < 0$, then $\tau_{nn} < 0$. Importantly, this condition is not an extra equality-type constraint on $\tau_{nn}$, meaning it is not a fixed parameter in our hydrodynamic model [30]. Notice that for bulk condensed matter systems the specific heat $\frac{\partial\epsilon}{\partial T}$ is generically expected to be positive.

A final, not necessarily independent, constraint on the relaxation time arises from requiring the linear stability of the modes. In $d+1$ dimensions there are $d+2$ modes, which at zero-wave vector [31] are

$$\omega = -\frac{i}{\tau_p},$$
$$\omega = -\frac{i}{2}\left(\frac{1}{\tau_{\epsilon\epsilon}} + \frac{1}{\tau_{nn}}\right) \pm \frac{i}{2}\sqrt{\left(\frac{1}{\tau_{\epsilon\epsilon}} - \frac{1}{\tau_{nn}}\right)^2 + \frac{4}{\tau_{n\epsilon}\tau_{\epsilon n}}}, \quad (20)$$

where the first mode has multiplicity $d$. The stability of the state then requires that the imaginary part of these modes be negative. Thus the first mode simply gives us the same constraint, $\tau_p \geq 0$, imposed by enforcing the second law. If we liberate ourselves from the second law, but maintain Onsager reciprocity, we have three free parameters and can readily arrange for propagating and/or unstable modes from the second expression (20). On the other hand, employing all our constraints, we find that the second set of modes in (20) saturate the linearized stability requirement, i.e.,

$$\omega = 0, \qquad \omega = -i\left(\frac{1}{\tau_{nn}} + \frac{1}{\tau_{\epsilon\epsilon}}\right) = -\frac{i}{\tau_{nn}}\left(\frac{\frac{\partial\epsilon}{\partial T} - \mu_{(0)}\frac{\partial n}{\partial T}}{\frac{\partial\epsilon}{\partial T}}\right). \quad (21)$$

The second expression above gives another constraint on the sign of $\tau_{nn}$ that depends on the thermodynamics, similar to what happens in (19).

While we have determined the one parameter family of relaxations leading to a linearized hydrodynamics respecting Onsager reciprocity and the second law, we should remind ourselves that hydrodynamics is not just a linearized theory. While a full investigation of nonlinear corrections is beyond the scope of this paper, it is a reasonable question to ask whether our one-parameter linearized expressions can come from a nonlinear formulation. This boils down to a question of integrability in thermodynamics. To investigate it, let us change basis for our relaxations so that

$$\frac{1}{\tau_{\epsilon\epsilon}}\delta\epsilon + \frac{1}{\tau_{\epsilon n}}\delta n = \frac{1}{\tau_{\epsilon T}}\delta T + \frac{1}{\tau_{\epsilon\mu}}\delta\mu,$$
$$\frac{1}{\tau_{n\epsilon}}\delta\epsilon + \frac{1}{\tau_{nn}}\delta n = \frac{1}{\tau_{nT}}\delta T + \frac{1}{\tau_{n\mu}}\delta\mu. \quad (22)$$

Imposing our Onsager reciprocity constraint and the second law on our relaxation terms, we find

$$\frac{1}{\tau_{\epsilon T}} = 0, \qquad \frac{1}{\tau_{nT}} = 0, \qquad \tau_{\epsilon\mu} = \frac{\tau_{nn}}{\mu_{(0)}}\frac{\frac{\partial\epsilon}{\partial T}}{\frac{\partial n}{\partial\mu}\frac{\partial\epsilon}{\partial T} - \frac{\partial n}{\partial T}\frac{\partial\epsilon}{\partial\mu}},$$
$$\tau_{n\mu} = \frac{\tau_{nn}\frac{\partial\epsilon}{\partial T}}{\frac{\partial n}{\partial\mu}\frac{\partial\epsilon}{\partial T} - \frac{\partial n}{\partial T}\frac{\partial\epsilon}{\partial\mu}}. \quad (23)$$

Suppose there exists a pair of differentiable functions $\Gamma_\epsilon$ and $\Gamma_n$ at the nonlinear level whose linearizations lead to our relaxation terms,

$$\delta\Gamma_\epsilon = \frac{\partial\Gamma_\epsilon}{\partial\mu}\delta\mu + \frac{\partial\Gamma_\epsilon}{\partial\mu}\delta T = \frac{1}{\tau_{\epsilon\mu}}\delta\mu + \frac{1}{\tau_{\epsilon T}}\delta T, \quad (24a)$$

$$\delta\Gamma_n = \frac{\partial\Gamma_n}{\partial\mu}\delta\mu + \frac{\partial\Gamma_n}{\partial\mu}\delta T = \frac{1}{\tau_{n\mu}}\delta\mu + \frac{1}{\tau_{nT}}\delta T. \quad (24b)$$

It follows that our linearized relaxations are required to satisfy the integrability conditions,

$$\frac{\partial}{\partial T}\left(\frac{1}{\tau_{\epsilon\mu}}\right) = \frac{\partial}{\partial\mu}\left(\frac{1}{\tau_{\epsilon T}}\right), \qquad \frac{\partial}{\partial T}\left(\frac{1}{\tau_{n\mu}}\right) = \frac{\partial}{\partial\mu}\left(\frac{1}{\tau_{nT}}\right), \quad (25)$$

which come from commutativity of second derivatives on $\Gamma_\epsilon$ and $\Gamma_n$ [32]. From (23) we see that

$$\frac{\partial}{\partial\mu}\left(\frac{1}{\tau_{\epsilon T}}\right) = 0, \qquad \frac{\partial}{\partial\mu}\left(\frac{1}{\tau_{nT}}\right) = 0, \quad (26)$$

and plugging this result into (25), we find $\tau_{\epsilon\mu}$ and $\tau_{n\mu}$ are independent of $T$. Therefore, the most general $\tau_{nn}$ compatible with the existence of $\Gamma_\epsilon$ and $\Gamma_n$ are

$$\frac{1}{\tau_{nn}(T,\mu)} = \frac{f(\mu)\frac{\partial\epsilon}{\partial T}}{\frac{\partial n}{\partial\mu}\frac{\partial\epsilon}{\partial T} - \frac{\partial n}{\partial T}\frac{\partial\epsilon}{\partial\mu}}, \quad (27)$$

for $f$ an arbitrary function of $\mu$, while all other relaxations are fixed by our constraints. We repeat that this final result is a consequence of Onsager reciprocity, positivity of entropy production, and the existence of a nonlinear lift of the linearized quasihydrodynamic model. Sacrificing even one of the requirements leads to a much more general result.

To conclude this section, we comment that the fluid with relaxations described above is quite different from the one presented in [15]. There we considered a complete (nonlinear) theory of hydrodynamics, in the hydrostatic regime, while the relaxations considered here are defined to be out-of-equilibrium quantities. Moreover, the constraint obtained in [15] between energy $\Gamma_\epsilon$ and momentum relaxation $\Gamma_p$, i.e.,

$$\Gamma_\epsilon = \Gamma_p p^i v_i, \tag{28}$$

ensures that at linear order in small velocity, energy is still conserved and only momentum decays with the usual constraint $\tau_p^{-1} = \Gamma_p \geq 0$.

## III. THE BACKGROUND FIELD METHOD AND TIME-REVERSAL COVARIANCE

In the previous section, we saw how imposing Onsager reciprocity on the correlators between conserved charges constrains the possible form of charge relaxation for an example charged relativistic fluid. In the present section, we explore whether additional constraints appear when we extend time-reversal covariance to the correlators between the full current and energy-momentum tensor.

To evaluate the complete correlators, we use the background field approach. In the background field approach, one places the fluid on a curved background $g_{\mu\nu}$ with a generic gauge field $A_\mu$ and defines the expectation values

$$\mathcal{J}^\mu(x) = \sqrt{-g}\langle J^\mu(x)\rangle_{A,g}, \qquad \mathcal{T}^{\mu\nu}(x) = \sqrt{-g}\langle T^{\mu\nu}(x)\rangle_{A,g}. \tag{29}$$

On the rhs of each equality, the expectation values of $J^\mu$ and $T^{\mu\nu}$ in the presence of $A_\mu$ and $g_{\mu\nu}$ are given by the on-shell values of $J^\mu$ and $T^{\mu\nu}$. This point of view allows us to define the retarded correlators of the stress tensor and current via varying with respect to the sources $A_\mu$ and $g_{\mu\nu}$ and then taking the flat space limit [16], i.e.,

$$\langle J^\mu J^\nu\rangle_{\mathrm{R}}(x) = -\frac{\delta\mathcal{J}^\mu(x)}{\delta A_\nu(0)}\bigg|_{g=\eta,A=0}, \qquad \langle T^{\mu\nu} J^\rho\rangle_{\mathrm{R}}(x) = -\frac{\delta\mathcal{T}^{\mu\nu}(x)}{\delta A_\rho(0)}\bigg|_{g=\eta,A=0}, \tag{30a}$$

$$\langle J^\mu T^{\nu\rho}\rangle_{\mathrm{R}}(x) = -2\frac{\delta\mathcal{J}^\mu(x)}{\delta g_{\nu\rho}(0)}\bigg|_{g=\eta,A=0}, \qquad \langle T^{\mu\nu} T^{\rho\sigma}\rangle_{\mathrm{R}}(x) = -2\frac{\delta\mathcal{T}^{\mu\nu}(x)}{\delta g_{\rho\sigma}(0)}\bigg|_{g=\eta,A=0}. \tag{30b}$$

We see that by solving the linearized hydrodynamic equations for the evolution of the hydrodynamic fields, in terms of the external sources, this method gives direct access to all the correlators of the stress tensor and charge current.

We consider now the same theory of linearized hydrodynamics discussed in the previous section, but this time placed on a curved background. The nonlinear Landau-frame constitutive relations are, up to order one in derivatives,

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + P\Delta^{\mu\nu} - \eta\Delta^{\mu\alpha}\Delta^{\nu\beta}\left(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha - \frac{2}{d}g_{\alpha\beta}\nabla_\lambda u^\lambda\right) - \zeta\nabla_\mu u^\mu + \mathcal{O}(\nabla^2), \tag{31a}$$

$$J^\mu = n u^\mu + \sigma\Delta^{\mu\nu}\left(E_\nu - T\nabla_\nu\frac{\mu}{T}\right) + \mathcal{O}(\nabla^2), \tag{31b}$$

where $\Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$ is the projector transverse to the velocity and $E_\mu = F_{\mu\nu}u^\nu$ the electric field.

The equations of motion corresponding to these constitutive relations are simply the standard conservation equations of energy, momentum, and charge in the presence of curved metric and gauge fields, which we now need to supplement with relaxation terms. This is a nontrivial problem at the nonlinear level, since—even if one writes relaxations in terms of Lorentz covariant objects, such as $u_\mu J^\mu/\tau$ for the charge relaxation (see, e.g., [7]) or the momentum relaxation terms in [20]—relaxation terms, in general, break Lorentz covariance. Hence there are many ways to introduce relaxation terms at the nonlinear level which lead to the same linearized expressions (8), and non-Lorentz-covariant nonlinear expressions cannot be written down uniquely, in general. We can, however, write relaxation terms in a covariant-looking form by introducing external vectors into the theory to contract the appropriate indices. For example, we can introduce a fixed timelike vector which picks out the frame in which the charges obtain their stationary values. By choosing this vector to be dual to the clock form $\tau^\mu = (1, \vec{0})$ as in [33–37], we pick out the lab frame as the frame where the charges become stationary and can write, e.g., the charge relaxation term $J^t/\tau = \tau_\mu J^\mu/\tau$ in a covariant-looking way [38]. With this construction it is possible to covariantize the relaxation terms in a weak sense, but Lorentz covariance is still lost.

To make the above point clearer, we write here a possible representative set of relaxation terms which reduce to (8) when linearizing around flat spacetime. Notice that while $\tau_\mu(x)$ is coordinate covariant, in Cartesian coordinates in $(d+1)$-dimensional Minkowski space we can choose for it to be $\tau_\mu = (-1, \vec{0})$. Consequently, $\nabla^{(0)}_\mu \tau^\mu = 0$ in all coordinate systems parametrizing Minkowski space. Given this, one possible way to write the nonlinear equations of motion is

$$\nabla_\mu T^{\mu\nu} = F^{\nu\lambda} J_\lambda + \frac{1}{\tau_{\epsilon n}} \tau^\nu (J\cdot\tau) - \frac{1}{\tau_{\epsilon\epsilon}} \tau^\nu (T^{\alpha\beta}\tau_\alpha\tau_\beta) + \frac{1}{\tau_m}(g^{\nu\alpha} + \tau^\nu\tau^\alpha) T_{\alpha\beta}\tau^\beta, \tag{32a}$$

$$\nabla_\mu J^\mu = \frac{1}{\tau_{nn}} J^\mu \tau_\mu - \frac{1}{\tau_{n\epsilon}} T^{\mu\nu}\tau_\nu\tau_\mu. \tag{32b}$$

In the above expressions one can replace any $\tau^\mu$ with a $u^\mu$. These various choices of nonlinear extensions represent a space of distinct theories as can be readily argued by considering the value of the relaxation terms when the fluid is moving with some spatial velocity. Nevertheless, they lead to the same linearized hydrodynamics around a zero spatial velocity background.

With the above discussion in mind, to avoid the issue of nonlinear extensions entirely, we linearize the equations of motion corresponding to (31) around an equilibrium configuration with constant temperature, constant chemical potential, zero fluid velocity, and flat spacetime,

$$T = T_{(0)} + \delta T, \qquad \mu = \mu_{(0)} + \delta\mu, \qquad u^\mu = \tau^\mu + \delta v^\mu, \tag{33a}$$

$$g^{\mu\nu} = \eta^{\mu\nu} + \delta h^{\mu\nu}, \qquad A^\mu = \delta A^\mu. \tag{33b}$$

Eventually, the (linearized) equations of motion read

$$\tau^\mu \partial_\mu \delta\epsilon + (\epsilon_{(0)} + P_{(0)}) \nabla^{(0)}_\mu \delta v^\mu = -\left(\frac{1}{\tau_{\epsilon\epsilon}}\delta\epsilon + \frac{1}{\tau_{\epsilon n}}\delta n\right), \tag{34a}$$

$$\tau^\mu \partial_\mu \delta n + n_{(0)} \nabla^{(0)}_\mu \delta v^\mu - \sigma\left(P^{\mu\nu}\nabla^{(0)}_\mu \partial_\nu \delta\mu - \frac{\mu_{(0)}}{T_{(0)}} P^{\mu\nu}\nabla^{(0)}_\mu \partial_\nu \delta T - \nabla^{(0)}_\mu \delta E^\mu\right) = -\left(\frac{1}{\tau_{n\epsilon}}\delta\epsilon + \frac{1}{\tau_{nn}}\delta n\right), \tag{34b}$$

$$\begin{aligned} P^{\mu\nu}\partial_\nu \delta P - \zeta P^{\mu\nu}\nabla^{(0)}_\nu \nabla^{(0)}_\rho \delta v^\rho + (\epsilon_{(0)} + P_{(0)}) P^\mu{}_\rho (\tau^\nu \nabla^{(0)}_\nu \delta v^\rho) - 2\eta P^{\mu\rho}\nabla^{(0)}_\sigma \delta\sigma^\sigma{}_\rho = &-\frac{1}{\tau_p}(\epsilon_{(0)} + P_{(0)})\delta v^\mu + n_{(0)}\delta E^\mu \\ &- P_{(0)} P^\mu{}_\nu \nabla^{(0)}_\rho \delta h^{\rho\nu} - (\epsilon_{(0)} + P_{(0)}) \\ &\times P^\mu{}_\rho \tau^\nu \delta\Gamma^\rho_{\nu\sigma}\tau^\sigma, \end{aligned} \tag{34c}$$

where $P^{\mu\nu} = \eta^{\mu\nu} + \tau^\mu\tau^\nu$ is the projector transverse to the background velocity and $\nabla^{(0)}_\mu$ the background connection defined in terms of the Christoffel symbols $\Gamma^\rho_{\nu\sigma}$. Note that we have explicitly ignored any term proportional to a product of a relaxation time and a gauge or metric fluctuation. The reason for doing so is that distinct nonlinear formulations can lead to different terms of this kind. For example, the nonlinear extension displayed in (32) leads to

$$\text{energy} \quad -\frac{\delta n}{\tau_{\epsilon n}} - \frac{\delta\epsilon}{\tau_{\epsilon\epsilon}} - \frac{1}{2}\delta h_{tt}\left(\frac{n_{(0)}}{\tau_{\epsilon n}} + \frac{\epsilon_{(0)}}{\tau_{\epsilon\epsilon}}\right), \tag{35a}$$

$$\text{momentum} \quad -\frac{\epsilon_{(0)} + P_{(0)}}{\tau_p}\delta v^i, \tag{35b}$$

$$\text{charge} \quad -\frac{\delta n}{\tau_{nn}} - \frac{\delta\epsilon}{\tau_{n\epsilon}}, \tag{35c}$$

where the final term in (35a) is absent from (34). In addition, replacing some of the $\tau^\mu$ in (32) with $u^\mu$ leads to different source terms in our linearized hydrodynamic equations. We argue below that this is not an issue, as enforcing Onsager reciprocity requires us to modify these terms by hand anyway [see Eq. (37)]. Hence, we may take (34) as our base case without loss of generality. Finally, we emphasize that the equations (34) are made explicitly coordinate invariant by the use of $\nabla^{(0)}_\mu$ instead of $\partial_\mu$ for the background covariant derivative, which accounts for the existence of nonlinearized connections on a flat background in a curvilinear coordinate system.

Given the above setup, it is not difficult to show that there are correlators derived from (34) which do not satisfy the time-reversal covariance condition (4). For example,

$$\langle T^{tt}T^{xx}\rangle - \langle T^{xx}T^{tt}\rangle|_{\mathbf{k}=0} = -\frac{((\epsilon_{(0)}+P_{(0)})(\tau_{n\epsilon}\tau_{\epsilon n}-\tau_{nn}\tau_{\epsilon\epsilon})) + i\tau_{nn}\tau_{n\epsilon}((\epsilon_{(0)}+P_{(0)})\tau_{\epsilon n}+n_{(0)}\tau_{\epsilon\epsilon})\omega}{\tau_{nn}\tau_{\epsilon\epsilon}+\tau_{n\epsilon}\tau_{\epsilon n}(i+\tau_{nn}\omega)(i+\tau_{\epsilon\epsilon}\omega)}. \tag{36}$$

This remains the case, even when we identify the energy and charge relaxation terms with those preserving the Onsager relations (and the second law) in the Martin-Kadanoff approach, (11). In addition, the correlators are generically different from the ones obtained in the previous section [39].

If we want the correlators to be time-reversal covariant, then we must modify some aspect of our hydrodynamic formulation. We choose to modify the hydrodynamic equations, by including additional source terms in the hydrodynamic equations that vanish when the metric and gauge field take their background values. That is, we include additional sources to the hydrodynamic equations depending nontrivially on the differences $\delta(g_{\mu\nu}-\eta_{\mu\nu})$ and $\delta(A_\mu - A^{(0)}_\mu)$. The relaxations can then be understood as explicitly breaking the background independence of the theory, which manifests in a preferred metric and gauge field. This is not to say that one cannot place this relaxed hydrodynamics on a curved background, only that the equation of motion depends explicitly on the background metric. It is also important to reiterate that the resultant theory is coordinate invariant, even if it is not background independent.

To employ our method, we proceed by brute force and compute the correlators with arbitrary additional source terms constructed from $\delta h_{\mu\nu}$ and $\delta A_\mu$. In particular, we schematically write [40]

$$(\text{sources of }(34)) \to (\text{sources of }(34)) + c_\alpha{}^{\mu\nu}\delta h_{\mu\nu} + r_\alpha{}^\mu \delta A_\mu \tag{37}$$

and compute the two point functions using these modified equations of motion via the variational approach. In the above expression $\alpha = \{\epsilon, n, x, y, z\}$ identifies the relevant equation of motion. Imposing time-reversal covariance on these Green's functions gives us a set of relations that must be solved for the arbitrary coefficients in (37). We do this for the full correlator obtained from the background field method, rather than the truncated correlator (where one accounts for and excises any spurious higher derivative terms). Consequently, our complete correlators respect time-reversal covariance.

There are in total 70 source terms that we can add, however, since there are no parity-breaking operators in the theory, $c_\alpha{}^{\mu\nu}$, $r_\alpha{}^\mu$ are parity even and this allows us to use rotational invariance and parity with respect to a single axis (which itself is a consequence of parity and rotational invariance) $\mathcal{P}_i$: $i \to -i$ ($i = x, y, z$) to reduce the number of sources to only 13. Because $\tau_p$ has the same value in all the directions, we also make the ansatz that the sources are isotropic, which reduces their number down to 9. Finally, after imposing time-reversal covariance [41] on the full correlators we end up with only 4 new source terms, while all the other coefficients are found to be zero. The only equations of motion that receive corrections are those for energy and charge, i.e.,

$$\text{energy:}\quad -\left(\frac{1}{\tau_{\epsilon\epsilon}}\delta\epsilon + \frac{1}{\tau_{\epsilon n}}\delta n\right) - c_\epsilon{}^{tt}\delta h_{tt} - r_\epsilon{}^t \delta A_t, \tag{38a}$$

$$\text{charge:}\quad -\left(\frac{1}{\tau_{n\epsilon}}\delta\epsilon + \frac{1}{\tau_{nn}}\delta n\right) - c_n{}^{tt}\delta h_{tt} - r_n{}^t \delta A_t, \tag{38b}$$

where the value of the coefficients is fixed in terms of the standard relaxations,

$$c_\epsilon{}^{tt} = \frac{1}{2}\left(\frac{\chi_{\epsilon n}}{\tau_{\epsilon n}} + \frac{\chi_{\epsilon\epsilon}}{\tau_{\epsilon\epsilon}}\right), \tag{39a}$$

$$r_\epsilon{}^t = \frac{\chi_{\epsilon\epsilon}}{\tau_{n\epsilon}} + \frac{\chi_{\epsilon n}}{\tau_{nn}} = 2c_n{}^{tt}, \tag{39b}$$

$$c_n{}^{tt} = \frac{1}{2}\left(\frac{\chi_{\epsilon\epsilon}}{\tau_{n\epsilon}} + \frac{\chi_{\epsilon n}}{\tau_{nn}}\right), \tag{39c}$$

$$r_n{}^t = \frac{\chi_{\epsilon n}}{\tau_{n\epsilon}} + \frac{\chi_{nn}}{\tau_{nn}}. \tag{39d}$$

There are no additional constraints on the relaxation time parameters $\tau$ beyond those imposed in the Martin-Kadanoff procedure (11). Hence we have confirmed that (i) relaxed hydrodynamics can be made time-reversal covariant and (ii) that the correlators obtained by the variational procedure including the $c$'s and $r$'s agree with those obtained by the Martin-Kadanoff one, up to the usual contact terms [42]. As the same exact procedure presented above can be applied to a perfect fluid, leading to the same results (39), our expressions are hydrodynamic frame covariant as one would expect of a physically meaningful theory.

Because we are interested in the set of all correlators of the theory, knowing the linearized expressions for a specific background is as good as knowing the full theory, this is why our starting point was (34). Clearly, picking a different background (curved manifold, background gauge fields, nonzero spatial velocity,…), or working with a specific nonlinear form for the relaxation rates (e.g., $u_\mu J^\mu/\tau$ compared to $J^t/\tau$) will change the values of the coefficients $c$ and $r$ in (39). Nevertheless, we expect the procedure we suggest (add all possible source terms and impose Onsager relations) to hold for all different backgrounds.

Finally, one can reconsider positivity of entropy production at linear order in fluctuations in light of our new metric and gauge field fluctuation terms. On a weakly curved background the equivalent expression to (14) is

$$T_{(0)}\nabla^{(0)}_\mu \delta s^\mu = \delta\epsilon\left(\frac{\mu_{(0)}}{\tau_{n\epsilon}} - \frac{1}{\tau_{\epsilon\epsilon}}\right) + \delta n\left(\frac{\mu_{(0)}}{\tau_{nn}} - \frac{1}{\tau_{\epsilon n}}\right), \tag{40}$$

$$- ({r_\epsilon}^t - \mu_{(0)}{r_n}^t)\tau^\mu \delta A_\mu - ({c_\epsilon}^{tt} - \mu_{(0)}{c_n}^{tt}) \times \tau^\mu \delta g_{\mu\nu}\tau^\nu + \mathcal{O}(\delta^2, \partial^2). \tag{41}$$

As we have not added second order in fluctuation additional terms, this is the only expression we need to consider. Somewhat miraculously, imposing the entropy positivity conditions we found in Sec. II, (15) and (9), also happens to set the new terms to zero. Thus, our inclusion of background-dependent terms to the energy and charge conservation equations does not violate the second law to order one in fluctuations.

## IV. DISCUSSION

In this paper we have analyzed the implications of time-reversal covariance of the microscopic theory, i.e., Onsager-Casimir relations, on a theory of linearized relativistic hydrodynamics in the presence of generic relaxations. We found a set of constraints that the relaxation parameters must obey in order for the fluid to satisfy Onsager relations (11), positivity of entropy production (9), and linearized stability (20), which reduce the number of free relaxation parameters to only one.

Subsequently, we have computed all the retarded two-point functions for such a fluid by considering small perturbations of the metric and gauge field: we found that, in general, this method gives Green's functions that are not time-reversal covariant and do not match with the ones obtained by linear response theory. One of the core results of this work is to show that it is possible to overcome these problems by considering extra source terms in the equations of motion and, surprisingly, the coefficients of these source terms are completely fixed by the simple requirement of time-reversal covariance, leading to the final result (39).

Although we tested this procedure only on a relativistic fluid with a $U(1)$ symmetry, which is the simplest one to couple to curved spacetime, we expect the same method to also work for other hydrodynamic theories with different spacetime and internal symmetries. All the more so as our constraints can be derived at the ideal level. Furthermore, this approach should work also for nontrivial equilibrium backgrounds, e.g., a constant magnetic field, a curved spacetime, or in the presence of topological terms such as Chern-Simon's terms [44,45]. To check the validity of this claim could be the goal of succeeding works, and it would be quite interesting to find a situation where time-reversal covariance and positivity of entropy production are not sufficient to fix the extra variational terms.

Regarding future perspectives, it would be interesting to study how these relaxations can be consistently included in the quasihydrodynamic description beyond the linearized regime. In particular, the presence of relaxation terms in the equations of motion could induce new transport coefficients in the constitutive relations or modify the values of known ones. One could also consider hydrodynamic $N$-point functions and ascertain whether time-reversal covariance is sufficient to fix higher order in fluctuation terms in the effective hydrodynamic equations.

It is also possible, with the findings of this work, to reanalyze certain known results related to the transport properties of anomalous hydrodynamics, e.g., in the context of studying the thermoelectric properties of Weyl semimetals. Specifically, as already mentioned in the Introduction, generic relaxations are needed to obtain finite DC conductivities for an anomalous fluid [7–9], hence it would be fruitful to apply the methods developed here to study the DC limit of the conductivities presented in [46].

Finally, it could be interesting to study models of relaxed hydrodynamics in the context of kinetic theory [12], holography [10], or using the Schwinger-Keldysh effective field theory (EFT) formalism [47]. Because these approaches often impose different constraints on the EFT compared to this work [48], they could give insight in order to check the universality of our results and how they are realized in different contexts.

## ACKNOWLEDGMENTS

We would like to acknowledge helpful discussions with Pavel Kovtun, Francisco Peña-Benitez, Amos Yarom, and

Vaios Ziogas. A. A. and I. M. have been partially supported by the "Curiosity Driven Grant 2020" of the University of Genoa and the INFN Scientific Initiative SFT: "Statistical Field Theory, Low-Dimensional Systems, Integrable Models and Applications." This project has also received funding from the European Union's Horizon 2020 research and innovation program under the Marie Skłodowska-Curie Grant Agreement No. 101030915.

[1] S. A. Hartnoll, P. K. Kovtun, M. Muller, and S. Sachdev, Theory of the Nernst effect near quantum phase transitions in condensed matter, and in dyonic black holes, Phys. Rev. B **76**, 144502 (2007).

[2] T. Andrade and B. Withers, A simple holographic model of momentum relaxation, J. High Energy Phys. 05 (2014) 101.

[3] A. Amoretti, A. Braggio, N. Maggiore, and N. Magnoli, Thermo-electric transport in gauge/gravity models, Adv. Phys. X **2**, 409 (2017).

[4] A. Amoretti, A. Braggio, G. Caruso, N. Maggiore, and N. Magnoli, Holography in flat spacetime: 4D theories and electromagnetic duality on the border, J. High Energy Phys. 04 (2014) 142.

[5] A. Amoretti, A. Braggio, N. Maggiore, N. Magnoli, and D. Musso, Thermo-electric transport in gauge/gravity models with momentum dissipation, J. High Energy Phys. 09 (2014) 160.

[6] A. Amoretti and D. Musso, Magneto-transport from momentum dissipating holography, J. High Energy Phys. 09 (2015) 094.

[7] K. Landsteiner, Y. Liu, and Y.-W. Sun, Negative magnetoresistivity in chiral fluids and holography, J. High Energy Phys. 03 (2015) 127.

[8] N. Abbasi, A. Ghazi, F. Taghinavaz, and O. Tavakol, Magneto-transport in an anomalous fluid with weakly broken symmetries, in weak and strong regime, J. High Energy Phys. 05 (2019) 206.

[9] A. Lucas, R. A. Davison, and S. Sachdev, Hydrodynamic theory of thermoelectric transport and negative magnetoresistance in Weyl semimetals, Proc. Natl. Acad. Sci. U.S.A. **113**, 9463 (2016).

[10] A. Jimenez-Alba, K. Landsteiner, Y. Liu, and Y.-W. Sun, Anomalous magnetoconductivity and relaxation times in holography, J. High Energy Phys. 07 (2015) 117.

[11] K. Pongsangangan, T. Ludwig, H. T. C. Stoof, and L. Fritz, Hydrodynamics of charged two-dimensional Dirac systems. I. Thermoelectric transport, Phys. Rev. B **106**, 205126 (2022).

[12] L. Fritz and T. Scaffidi, Hydrodynamic electronic transport, arXiv:2303.14205.

[13] B. N. Narozhny and I. V. Gornyi, Hydrodynamic approach to electronic transport in graphene: Energy relaxation, Front. Phys. **9**, 640649 (2021).

[14] V. Gall, B. N. Narozhny, and I. V. Gornyi, Electronic viscosity and energy relaxation in neutral graphene, Phys. Rev. B **107**, 045413 (2023).

[15] A. Amoretti, D. K. Brattan, L. Martinoia, and I. Matthaiakakis, Non-dissipative electrically driven fluids, J. High Energy Phys. 05 (2023) 218.

[16] P. Kovtun, Lectures on hydrodynamic fluctuations in relativistic theories, J. Phys. A **45**, 473001 (2012).

[17] L. P. Kadanoff and P. C. Martin, Hydrodynamic equations and correlation functions, Ann. Phys. (N.Y.) **24**, 419 (1963).

[18] Notice that it is not possible to simply swap each $\omega$ with $\omega + \frac{i}{\tau}$ in the Green's functions.

[19] L. V. Delacrétaz, B. Goutéraux, and V. Ziogas, Damping of Pseudo-Goldstone Fields, Phys. Rev. Lett. **128**, 141601 (2022).

[20] M. A. Stephanov and H.-U. Yee, No-Drag Frame for Anomalous Chiral Fluid, Phys. Rev. Lett. **116**, 122302 (2016).

[21] This is even more relevant in the presence of energy and charge relaxation because in this scenario we cannot naively use Ward identities to relate, e.g., $\langle J^t J^t \rangle$ to $\langle J^i J^i \rangle$.

[22] The even eigenvalue terms, i.e., $\eta_a = +1$, are $T$ and $\mu$ and functions of these variables. The odd terms are just $v^i$, the spatial velocity, with $\eta_{v^i} = -1$. All other eigenvalues can be determined from these quantities.

[23] Recall the susceptibility matrix is symmetric, $\chi_{ab} = \chi_{ba}$.

[24] Our relaxations will break Lorentz invariance and one may, in principle, expect new transport coefficients to appear depending on how the microscopic theory couples to processes responsible for breaking this symmetry. These new coefficients will not change results related to our relaxations, and we can, without loss of generality, assume that they happen to be zero for our fluid henceforth.

[25] We could also express the second law in terms of $\delta T$ and $\delta\mu$. This leads to a $2 \times 2$ linear system of equations that has a solution with $\det\chi = 0$. In what follows, we ignore this solution as unphysical.

[26] A. Amoretti, D. K. Brattan, N. Magnoli, and M. Scanavino, Magneto-thermal transport implies an incoherent Hall conductivity, J. High Energy Phys. 08 (2020) 097.

[27] A. Amoretti, D. Arean, D. K. Brattan, and N. Magnoli, Hydrodynamic magneto-transport in charge density wave states, J. High Energy Phys. 05 (2021) 027.

[28] A. Amoretti, D. Arean, D. K. Brattan, and L. Martinoia, Hydrodynamic magneto-transport in holographic charge density wave states, J. High Energy Phys. 11 (2021) 011.

[29] A. Amoretti and D. K. Brattan, On the hydrodynamics of $(2+1)$-dimensional strongly coupled relativistic theories in an external magnetic field, Mod. Phys. Lett. A **37**, 2230010 (2022).

[30] Moreover, one could add a second order in fluctuation piece proportional to $\delta\mu^2$ as a charge current or energy relaxation term. Subsequently, we can completely lift any constraints imposed on $\tau_{nn}$ by tuning the coefficient of this term appropriately.

[31] The expressions at nonzero wave vector are complicated and dependent on how one chooses to scale the relaxation terms in comparison to the momenta.
[32] It is clear that no equivalent $\Gamma_p$ can exist.
[33] J. de Boer, J. Hartong, N. A. Obers, W. Sybesma, and S. Vandoren, Perfect fluids, SciPost Phys. **5**, 003 (2018).
[34] J. de Boer, J. Hartong, N. A. Obers, W. Sybesma, and S. Vandoren, Hydrodynamic modes of homogeneous and isotropic fluids, SciPost Phys. **5**, 014 (2018).
[35] J. de Boer, J. Hartong, E. Have, N. A. Obers, and W. Sybesma, Non-boost invariant fluid dynamics, SciPost Phys. **9**, 018 (2020).
[36] I. Novak, J. Sonner, and B. Withers, Hydrodynamics without boosts, J. High Energy Phys. 07 (2020) 165.
[37] J. Armas, J. Hartong, E. Have, B. F. Nielsen, and N. A. Obers, Newton-Cartan submanifolds and fluid membranes, Phys. Rev. E **101**, 062803 (2020).
[38] This agrees with $u_\mu J^\mu/\tau$ only at zero spatial velocity but is generically different.
[39] Even after taking into account the fact that the two approaches may differ by contact terms.
[40] We have also confirmed if one includes relaxation terms that have explicit derivatives of the hydrodynamic variables, then one is also required to add derivatives of $\delta h_{\mu\nu}$ and $\delta A_\mu$ to the linearized equation of motion.
[41] On a practical level: First we imposed time-reversal covariance relations at $\mathbf{k} = 0$, then at $\omega = k_y = k_z = 0$. These two sets of constraints are enough to identify all the coefficients. We, subsequently, check explicitly that the full correlators satisfy time-reversal covariance.
[42] We can also compare our results to those detailed in the appendix "Coupling to external sources" of [19] (see also [43]). The authors compute the constitutive relations from the Schwinger-Keldysh formalism of relaxed hydrodynamics without a stress tensor but with a pseudo-Goldstone field (which we can set to zero without issue). They find the presence of an additional term proportional to the time component of the gauge field as we have added. This is unsurprising as the Schwinger-Keldysh formalism has time-reversal covariance and positivity of entropy production built in.
[43] J. Armas, A. Jain, and R. Lier, Approximate symmetries, pseudo-goldstones, and the second law of thermodynamics, arXiv:2112.14373.
[44] D. K. Brattan and G. Lifschytz, Holographic plasma and anyonic fluids, J. High Energy Phys. 02 (2014) 090.
[45] D. K. Brattan, A strongly coupled anyon material, J. High Energy Phys. 11 (2015) 214.
[46] A. Amoretti, D. K. Brattan, L. Martinoia, and I. Matthaiakakis, Leading order magnetic field dependence of conductivities in anomalous hydrodynamics, Phys. Rev. D **108**, 016003 (2023).
[47] P. Glorioso and H. Liu, Lectures on non-equilibrium effective field theories and fluctuating hydrodynamics, *Proc. Sci.*, TASI2017 (2018) 008 [arXiv:1805.09331].
[48] For example, the Schwinger-Keldysh formalism requires the dynamical Kubo–Martin–Schwinger condition and unitarity, which in turn automatically impose nonlinear Onsager relations and positivity of entropy production.