

Thermodynamic constraints and exact scaling exponents of flocking matter

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We use advances in the formalism of boost agnostic passive fluids to constrain transport in polar active fluids, which are subsequently described by the Toner-Tu equations. Acknowledging that the system fundamentally breaks boost symmetry, we compel what were previously entirely phenomenological parameters in the Toner-Tu model to satisfy precise relationships among themselves. Consequently, we propose a thermodynamic argument to determine the scalings of the transport coefficients under dynamical renormalization group flow given that the scaling of the noise correlator is exact, as has been supported numerically. These scalings perfectly agree with the results of recent state-of-the-art numerical simulation and experiments.

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I. INTRODUCTION

Classical thermodynamics is the only physical theory of universal content that within the framework of applicability of its basic concepts will never be overthrown. Every challenger to the supremacy of this framework has fallen before it; whether they be black holes or, as we argue using advances in the theory of boost agnostic fluids, polar active systems.

Active systems are often touted as out-of-equilibrium physics, where conventional equilibrium thermodynamic principles are expected to fall short [1]. These systems consist of self-propelled units, or active particles, that convert stored or ambient energy into directional movement. The interactions between active particles and their environment lead to highly coordinated collective motion and mechanical stresses. The continuous energy exchange with the surroundings and the intrinsic activity of the fluid's components drive these systems out of equilibrium, giving rise to unique behaviors such as pattern formation, nonequilibrium phase transitions, and novel mechanical responses. These phenomena are prevalent in a wide array of systems ranging from bacteria to liquid crystals.

More specifically, a prominent role in this field is played by polar active systems, which are composed of self-propelled particles that align their movement with their closest neighbors. These systems demonstrate collective behaviors such as flocking and dynamic pattern formation, providing valuable insights into biological phenomena and serving as a framework for understanding complex systems.

Among the various theoretical approaches aimed at describing these systems, special attention is given to the model first proposed in [2–5]. There, Toner and Tu put forward a phenomenological description of flocking behavior, drawing

inspiration from the equations governing liquid crystals. They formulate a set of equations for a dynamic velocity field \vec{v} and a conserved number density n , resembling the Navier-Stokes equation for passive fluids. The system's preferred background velocity is attained by coupling these equations to an external potential U for the velocity field, akin to the well-known Mexican Hat potential, which causes the system to undergo a spontaneous breaking of the rotational symmetry.

Unfortunately, despite the considerable success of the Toner-Tu model in elucidating the large-scale dynamics of polar active systems, the model itself lacks a clear microscopic derivation. This is a gap only partially addressed by kinetic theory derivations [6–10], which, however, always rely on Galilean invariance in the form of Ward identities, as we shall discuss.

Returning to the Toner-Tu model itself, we remind the reader that the essence of any effective description is contained in identifying the relevant effective variables (n and \vec{v}) and symmetries. In particular, we assume the system has translation invariance, which is only broken by the presence of the external potential U , but lacks a boost symmetry. Then, each velocity must be treated as a distinct configuration of the system, as opposed to Galilean or Lorentzian models where boosts relate systems at different velocities.

The formalism for describing fluids without boost invariance is “boost agnostic hydrodynamics” [11,12]. For such fluids, the velocity appears as a variable in the global thermodynamic description of the system—just like the temperature and chemical potential. The boost agnostic formalism, as per the name, does not discern between the mechanisms that lead to the velocity being necessary to specify equilibrium. Subsequently, we can add a constraint force, represented by U , directly to our equations of motion to pick among the space of distinct velocities. As is typical in mechanics, such a constraint force has no consequences for the mechanical behavior of the system, which in our case is the thermodynamic dependence of the transport coefficients. More importantly, adding such a force to the effective equations breaks

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translation invariance in a precise manner as one might expect from a theory with nearest-neighbor interactions.

In this paper we present two results. First, we demonstrate that polar active fluids are akin to passive fluids lacking boost symmetry. This is opposed to some lore in the literature where the active fluid is completely divorced from the passive analog. For hydrodynamics to offer a sensible description [13], we must always assume that there is a scale separation within the system, ensuring rapid local equilibration compared to long-scale collective dynamics. The necessity of a local Gibbs-like distribution heavily constrains the constitutive relations of the ideal-fluid model. Notably, this implies that the quantity akin to pressure in the Toner-Tu equations can be treated as a thermodynamic pressure in many respects. Our argument centers on the requirement that an active fluid maintains consistency with basic thermodynamics, particularly in its disordered phase [14]. While tuning the symmetry-breaking potential restricts the space of steady-state configurations and consequently affects collective dynamics and response to perturbations [2–4], it does not alter the necessity for constitutive relations to adhere to thermodynamic principles. This argument is strongly supported by recent works, suggesting that standard hydrodynamics is a good description of systems that are in nonequilibrium steady states [15–18].

Secondly, these constraints enable us to derive analytical expressions for the critical exponents of the system in arbitrary spatial dimension d , aligning precisely with numerical computations in the Vicsek model [19] and differing from those derived by [3]. Our argument is grounded in the thermodynamic consistency of the ideal fluid constitutive relations, specifically the statement that at lowest order in derivatives the fluid should not produce entropy, and in principle can be applied to many different systems beyond the case of compressible polar fluids analyzed in this work.

II. THERMODYNAMICALLY CONSISTENT POLAR ACTIVE FLUIDS

In this and the next section we will show that it is possible to recover all the observable features of the phenomenological Toner and Tu model from the hydrodynamics of a simple passive fluid.

As our starting point, consider a simple fluid without boost symmetry. The relevant hydrodynamic variables are the chemical potential μ , associated with a conserved charge n , and the fluid velocity \vec{v} , which itself acts as a chemical potential for the fluid momentum \vec{g} . Following [12,20] we can define a generating functional for such fluid that produces the constitutive relations for the conserved currents in terms of a gradient expansion of the macroscopic variables. The leading term in a small derivative expansion for the effective action is then just the integral of a scalar, $P(\mu, \vec{v}^2)$, where P is the usual thermodynamic pressure. Variations of the background on which the fluid sits produce the desired constitutive relations. In particular, the ideal-fluid constitutive relations take the form [11,12,16,20]

$$g^i = \rho v^i, \quad \tau^{ij} = \rho v^i v^j + P \delta^{ij}, \quad j^i = n v^i, \quad (1a)$$

where $\vec{g} = (\partial P / \partial \vec{v})_\mu$ is the momentum conjugate to the velocity, $\rho = 2(\partial P / \partial \vec{v}^2)_\mu$ is the kinetic mass density, and $n = (\partial P / \partial \mu)_{v^2}$. In the absence of external forces, they satisfy the following conservation equations:

$$\partial_t g^i + \partial_j \tau^{ji} = 0, \quad \partial_t n + \partial_i j^i = 0. \quad (1b)$$

The conservation laws of (1b) are respectively associated with translation invariance and particle number. Notice, in particular, that $\vec{j} \neq \vec{g}$, which is a consequence of the breaking of the Milne or Galilean Ward identity that holds only for systems with an exact boost symmetry [21,22].

The ideal constitutive relations (1a) are the first terms in the derivative expansion corresponding to localizing our macroscopic variables. The next terms in the derivative expansion, correcting τ^{ij} and j^i in (1), lead to dissipative flows. This order-one hydrodynamics (for fluids without boost symmetry) has been studied in [12,20,23] and for ease of reference, we summarize the relevant expressions in Appendix A. After obtaining the order-one corrections, to complete the identification of our effective theory with the observed physics of polar active fluids, we turn on nonthermal noise and the external potential U in the normative manner. In particular, we add them to (1b) by hand. In this regard we treat U no differently to the noise \vec{f} in assuming that the form of the constitutive relations are unchanged by these external forces. The result is that \vec{g} is no longer a conserved quantity but it can nevertheless be a relevant hydrodynamic variable if the external potential, which breaks the symmetries, is weak enough [24,25]. In this sense U should be understood as imposing extra constraints on the system which reduce the allowed space of stationary states from those permitted to the passive fluid, i.e., the stationary state velocity will be fixed to a particular value.

Following these considerations, we state the Toner-Tu equations:

$$\begin{aligned} \partial_t \vec{v} + \lambda_1 (\vec{v} \cdot \vec{\nabla}) \vec{v} + \lambda_2 (\vec{\nabla} \cdot \vec{v}) \vec{v} + \lambda_3 \vec{\nabla} |\vec{v}|^2 + \vec{\nabla} P_1 \\ + \vec{v} (\vec{v} \cdot \vec{\nabla} P_2) = U \vec{v} + D_B \vec{\nabla} (\vec{\nabla} \cdot \vec{v}) \\ + D_T \nabla^2 \vec{v} + D_2 (\vec{v} \cdot \vec{\nabla})^2 \vec{v} + \vec{f}, \end{aligned} \quad (2a)$$

$$\partial_t n + \lambda_n \vec{\nabla} \cdot (n \vec{v}) = 0, \quad (2b)$$

where all transport coefficients above are functions of n and \vec{v}^2 , U is bounded below and has a zero at some \vec{v}^2 , and we assume that \vec{f} is a Gaussian white nonthermal noise

$$\langle f_i(\vec{x}, t) f_j(\vec{x}', t') \rangle = \Delta \delta_{ij} \delta^d(\vec{x} - \vec{x}') \delta(t - t') \quad (2c)$$

with Δ some constant. As was done in the original paper [3], we have suppressed derivatives of the particle density in (2a) for legibility. These are easily restored and do not affect the physics that we will be interested in. Moreover, to match the Toner-Tu equations we have chosen the relaxation term in (2a) to be $U \vec{v}$, ignoring the types of derivative corrections one can encounter in [26]. This represents a minimal choice for the constitutive relation of the relaxation term which must be set externally to the problem, as was done in the original papers.

The Toner-Tu equations above (2) include λ_1 , λ_2 , λ_3 , and λ_n as phenomenological parameters. Our ideal charge conser-

¹We set the particles' mass $m = 1$ without loss of generality.

vation equation agrees with the Toner and Tu model (2) upon setting $\lambda_n = 1$. At first glance, the same cannot be said about our ideal momentum equation, which from (1a) we write as

$$\partial_t \vec{v} + \vec{v}(\vec{\nabla} \cdot \vec{v}) + (\vec{v} \cdot \vec{\nabla})\vec{v} + \frac{\vec{\nabla} P}{\rho} + \frac{\vec{v}}{\rho} \partial_t \rho + \frac{\vec{v}}{\rho} \vec{v} \cdot \vec{\nabla} \rho = 0. \quad (3)$$

Using the number conservation equation we can express the last two terms above as

$$\partial_t \rho + \vec{v} \cdot \vec{\nabla} \rho = - \left(\frac{\partial \rho}{\partial n} \right)_{v^2} n \vec{\nabla} \cdot \vec{v} + \left(\frac{\partial \rho}{\partial v^2} \right)_n (\partial_t v^2 + \vec{v} \cdot \vec{\nabla} v^2). \quad (4)$$

Plugging this expression back into (3) and projecting the momentum equation along \vec{v} we can solve for $\partial_t v^2$. Finally, substituting the solution for $\partial_t v^2$ again into the momentum conservation equation we arrive at the final result

$$\begin{aligned} \partial_t \vec{v} + \lambda_1 (\vec{v} \cdot \vec{\nabla})\vec{v} + \lambda_2 \vec{v}(\vec{\nabla} \cdot \vec{v}) + \lambda_3 \vec{\nabla} |\vec{v}|^2 \\ + \frac{\vec{\nabla} P}{\rho} - k \frac{\vec{v}}{\rho^2} (\vec{v} \cdot \vec{\nabla} P) = 0, \end{aligned} \quad (5)$$

where the values of the parameters are

$$\lambda_1 = 1, \quad \lambda_2 = \frac{1 - \frac{\partial \rho}{\partial n} \frac{n}{\rho}}{1 + 2 \frac{\partial \rho}{\partial v^2} \frac{v^2}{\rho}}, \quad (6a)$$

$$\lambda_3 = 0, \quad k = 2 \frac{\partial \rho}{\partial v^2} \frac{1}{1 + 2 \frac{\partial \rho}{\partial v^2} \frac{v^2}{\rho}}, \quad (6b)$$

and the thermodynamic derivatives $\frac{\partial \rho}{\partial v^2}$ and $\frac{\partial \rho}{\partial n}$ are to be understood at constant n or v^2 , respectively. The last two terms in Eq. (5) can be identified as the transverse and longitudinal pressures P_1 and P_2 :

$$\vec{\nabla} P_1 = \frac{1}{\rho} \vec{\nabla} P, \quad \vec{v} \cdot \vec{\nabla} P_2 = \frac{\frac{2}{\rho} \frac{\partial \rho}{\partial v^2}}{1 + 2 \frac{\partial \rho}{\partial v^2} \frac{v^2}{\rho}} \vec{v} \cdot \vec{\nabla} P. \quad (6c)$$

We see that an anisotropic pressure term is a natural consequence of boost agnosticity even in equilibrium passive fluids.

Finally, we note that the coefficient λ_3 can be shifted to an arbitrary function. This happens because the pressure is a function of v^2 and can be expanded as

$$\frac{\vec{\nabla} P}{\rho} = \frac{1}{\rho} \left(\frac{\partial P}{\partial n} \right)_{v^2} \vec{\nabla} n + \frac{1}{\rho} \left(\frac{\partial P}{\partial v^2} \right)_n \vec{\nabla} |v|^2, \quad (7)$$

where the first term is interpreted as a new pressure gradient $\vec{\nabla} \tilde{P}$, while the second term defines $\lambda_3 = \frac{1}{\rho} \left(\frac{\partial P}{\partial v^2} \right)_n$. This was also the case in the original Toner-Tu model. We have made the logical choice for its value and assume that we are supplied with the form of the pressure P to fix any ambiguity.

Restoring the boost symmetry amounts to setting $\vec{g} = n\vec{v}$ in (6) as is done in some works [6–10]. This choice would force upon us the constraint $\lambda_1 = 1$ and $\lambda_2 = \lambda_3 = 0$, as can be seen by direct substitution into (6a) and (6b). Subsequently, one can redefine the chemical potential to remove the anisotropic pressure and general dependence of thermodynamics on v^2 . This conclusion is independent of the precise nature of the microscopic theory, following only from symmetries.

Interestingly, even without the imposition $\vec{g} = \vec{j}$, the requirement of having a local Gibbs-like distribution (i.e., a generating functional) has still heavily constrained the ideal fluid constitutive relations. Four *a priori* independent phenomenological parameters, $\lambda_1, \lambda_2, \lambda_3$, and λ_n from the original work [3] [see also (2)], are reduced to one (P) and its derivatives. That this constraint between transport coefficients exists even in the absence of boost invariance, when it had previously been understood to be a consequence of this symmetry, has not been appreciated until now.

III. THE ORDERED PHASE

Only for specific flows and choices of the potential U in (2) would we expect the system of equations to arrive at a state with zero background velocity $\vec{v} = \vec{0}$ at late times. For a v^2 -dependent generic potential, bounded below by zero, the system will flow to a zero of U and break rotations spontaneously. It is typical in the literature to take $U = \alpha - \beta v^2$, so that $U\vec{v}$ behaves like a derivative of the Mexican Hat potential. Around the new steady state we can then linearize in the velocity

$$\vec{v} = (v_0 + \delta v_{\parallel})\hat{x}_{\parallel} + \vec{v}_{\perp} \quad (8)$$

with δv_{\parallel} and \vec{v}_{\perp} small fluctuations which are respectively longitudinal or orthogonal to the background velocity $v_0\hat{x}_{\parallel}$. v_0 is defined such that it obeys $U(n_0, v_0^2) = 0$, with n_0 the equilibrium particle number density.

In the ordered phase the longitudinal mode associated with δv_{\parallel} is gapped, therefore it can be integrated out to find a set of reduced equations for the Goldstone-like gapless mode \vec{v}_{\perp} and $\delta n = n - n_0$. Our analysis is similar to that of [5]. In particular, we solve the equation for δv_{\parallel} iteratively in terms of \vec{v}_{\perp} and δn treating derivatives and linearization as small in the same parameter ε . On a practical level, this amounts to counting fluctuations and derivatives on a similar footing $O(\delta) \sim O(\partial) \sim O(\varepsilon)$, with ε a counting parameter. The equation of motion for the heavy mode δv_{\parallel} , up to $O(\varepsilon^2)$, reads

$$\begin{aligned} (\rho_0 + 2v_0^2 \rho_v) \partial_t \delta v_{\parallel} + \rho_0 v_0 (2\partial_{\parallel} \delta v_{\parallel} + \vec{\nabla}_{\perp} \cdot \vec{v}_{\perp}) \\ + (P_n + v_0^2 \rho_n) \partial_{\parallel} \delta n + 2v_0 \partial_{\parallel} \delta v_{\parallel} (P_v + v_0^2 \rho_v) \\ + v_0 \rho_n \partial_t \delta n = \delta U(v_0 + \delta v_{\parallel}) + O(\varepsilon^3), \end{aligned} \quad (9a)$$

where $\rho_0 = \rho(n_0, v_0^2)$ is the kinetic mass density computed on the background and we have introduced the notation

$$\left(\frac{\partial F}{\partial n} \right)_{v^2} = F_n, \quad \left(\frac{\partial F}{\partial v^2} \right)_n = F_v, \quad \left(\frac{\partial^2 F}{\partial n^2} \right)_{v^2} = F_{nn}, \quad (9b)$$

with $F = \{U, P, \rho\}$. These thermodynamic derivatives are evaluated on the background at $n = n_0$ and $v^2 = v_0^2$ too. Non-linearities appear in (18) from fluctuations of the external potential, which we expand as

$$\begin{aligned} \delta U = U_n \delta n + U_v [2v_0 \delta v_{\parallel} + (\delta v_{\parallel})^2 + |\vec{v}_{\perp}|^2] \\ + \frac{1}{2} U_{nn} (\delta n)^2 + O(\varepsilon^3). \end{aligned} \quad (9c)$$

We can now solve Eq. (9) iteratively for δv_{\parallel} in terms of \vec{v}_{\perp} and δn . At lowest order in fluctuations and derivatives the solution is simply given by

$$\delta v_{\parallel} \approx -\frac{U_n}{U_v} \frac{\delta n}{2v_0} + O(\varepsilon^2). \quad (10)$$

We can subsequently plug this approximate solution back into the derivative and nonlinear terms in (9) and solve again for δv_{\parallel} . The solution takes the general form

$$\begin{aligned} \delta v_{\parallel} = & -\alpha_1 \delta n - \alpha_2 (\delta n)^2 - \frac{|\vec{v}_{\perp}|^2}{2v_0} - \alpha_3 (\vec{\nabla} \cdot \vec{v}_{\perp}) \\ & - \alpha_4 \partial_{\parallel} \delta n + O(\varepsilon^3) \end{aligned} \quad (11)$$

where the coefficients are

$$\alpha_1 = \frac{1}{2v_0} \frac{U_n}{U_v}, \quad \alpha_2 = \frac{1}{8v_0^3} \frac{U_n^2}{U_v^2} + \frac{U_n^2}{4v_0}, \quad (12a)$$

$$\alpha_3 = \frac{\rho_n n_0}{2v_0 U_v} - \frac{\rho_0}{2v_0 U_v} - \frac{\rho_v n_0}{2v_0} \frac{U_n}{U_v^2} - \frac{n_0 \rho_0}{4v_0^3} \frac{U_n}{U_v^2}, \quad (12b)$$

$$\begin{aligned} \alpha_4 = & \frac{\rho_0}{4v_0^2} \frac{U_n}{U_v^2} - \frac{U_n}{U_v^2} \frac{\rho_n n_0}{4v_0^2} - \frac{P_n}{2v_0^2 U_v^2} + \frac{U_n}{U_v^2} \frac{P_v}{2v_0^2} \\ & + \frac{U_n^2}{U_v^3} \frac{\rho_v n_0}{4v_0^2} + \frac{U_n^2}{U_v^3} \frac{n_0 \rho_0}{8v_0^4}. \end{aligned} \quad (12c)$$

Subsequently eliminating δv_{\parallel} from the charge conservation equation we find, to $O(\varepsilon^3)$,

$$\begin{aligned} \partial_t \delta n + v_2 \partial_{\parallel} \delta n + n_0 \vec{\nabla}_{\perp} \cdot \vec{v}_{\perp} + w_1 \vec{\nabla}_{\perp} \cdot (\vec{v}_{\perp} \delta n) \\ - w_2 \partial_{\parallel} (\delta n)^2 - w_3 \partial_{\parallel} |\vec{v}_{\perp}|^2 - \xi_n \partial_{\parallel} (\vec{\nabla}_{\perp} \cdot \vec{v}_{\perp}) \\ - D_{\parallel} \partial_{\parallel}^2 \delta n - D_{\perp} \nabla_{\perp}^2 \delta n = 0, \end{aligned} \quad (13a)$$

where the precise expressions of the coefficients can be found in Appendix B. Similarly, the equation of motion for the transverse velocity fluctuations \vec{v}_{\perp} becomes

$$\begin{aligned} \partial_t \vec{v}_{\perp} + v_0 \partial_{\parallel} \vec{v}_{\perp} + \kappa \vec{\nabla}_{\perp} \delta n + g_1 \delta n \partial_{\parallel} \vec{v}_{\perp} + g_2 \vec{v}_{\perp} \partial_{\parallel} \delta n \\ + g_3 \vec{\nabla}_{\perp} (\delta n)^2 + g_4 \vec{v}_{\perp} (\vec{\nabla}_{\perp} \cdot \vec{v}_{\perp}) + g_5 (\vec{v}_{\perp} \cdot \vec{\nabla}_{\perp}) \vec{v}_{\perp} \\ - D_v \vec{\nabla}_{\perp} \partial_{\parallel} \delta n - \xi_v \vec{\nabla}_{\perp} (\vec{\nabla}_{\perp} \cdot \vec{v}_{\perp}) \\ - \xi_{\perp} \nabla_{\perp}^2 \vec{v}_{\perp} - \xi_{\parallel} \partial_{\parallel}^2 \vec{v}_{\perp} = \vec{f}_{\perp}. \end{aligned} \quad (13b)$$

The v_2 and κ terms are linear in fluctuations and derivatives, the g_i and w_i terms are the nonlinear corrections, while the generalized diffusivities D_i and viscosities ξ_i are related to first-order transport coefficients (see Appendix A).

The form of these reduced Eqs. (13a) and (13b), matches exactly the expressions in [5], with a few minor differences. First, we observe that we have a new nonlinear term g_4 , which we believe is incorrectly missing in [5] due to some approximation. Secondly, we find that the bare coefficient D_{\perp} is nonzero, while it appears to be zero in [5]. Despite being zero, the term appears in the analysis of [5] since it is generated under a dynamical renormalization group, hence the difference is only in the bare theory. Finally, $g_5 = 1$ and v_0 appears in front of $\partial_{\parallel} \vec{v}_{\perp}$, despite the lack of Galilean invariance. Clearly, these small differences are irrelevant for the subsequent analysis, which matches exactly the one of [5].

In particular, since the linearized equations are the same as in [5], the modes and correlators take the same form, even if the values of the coefficients are more constrained in our passive fluid. Thus, we see that the nonequilibrium features characterizing active matter are not relevant for the hydrodynamic description of transport: all observables, such as modes, speeds of sound and correlators, can be obtained from the hydrodynamics of a passive fluid coupled to an external potential U .

To see this better we analyze one example in particular, which is more sensitive to the specific values of the parameters. One of the key signatures of the Toner and Tu model is the fact that there are two speeds of sound $v_{\pm}(\theta)$, where θ is the angle between the wave vector \vec{q} and the background velocity. Specifically at $\theta = 0$ one finds $v_{+}(\theta = 0) = v_0$ and $v_{-}(\theta = 0) = v_0 \lambda_1$ [3]. This effect is observed in simulations [27] and experiments [28]. Crucially, it was used to provide an estimate of $\lambda_1 \approx 0.75$, which was interpreted as a signature of the breaking of Galilean boost symmetry. As we have shown, even for boost-agnostic fluids we find $(g_5 =) \lambda_1 = 1$. It would naively seem that our model fails to reproduce the observed phenomena; however, this is not correct. Indeed, at $\theta = 0$ we find that the sound speed once again has two values [5]:

$$v_{+}(\theta = 0) = v_0, \quad v_{-}(\theta = 0) = v_0 \left(1 - \frac{n_0 U_n}{2v_0^2 U_v}\right) \quad (14)$$

and the difference vanishes when $U_n = (\frac{\partial U}{\partial n})_{v^2} = 0$. The fact that $U_n \neq 0$ is what allows us to have two different speeds of sound at $\theta = 0$, despite having $\lambda_1 = 1$. We remark that at arbitrary θ we observe the same angular dependence discussed in [3].

Interestingly, the parameter U_n in (14) is key to the breaking of time-reversal invariance [29,30]. Notice that the Toner and Tu equations [3] break Onsager relations [31] not only in the coefficient U_n , but also through the λ_i (see also [32] for a discussion on time-reversal invariance in active-matter systems). This is very unusual from the point of view of hydrodynamics; usually in hydrodynamics the ideal fluid (which follows from local equilibrium) is always time-reversal invariant. Breaking of the Onsager relations only appears through viscous corrections or external forces. Contrary to this, the Toner and Tu model violates Onsager relations at lowest order in derivatives, therefore equilibrium (even in the disordered phase) is not time-reversal invariant.

We emphasize the conclusions of this section as follows: we are able to reproduce the phenomenology, for example, the modes and response functions, of the Toner-Tu model from the hydrodynamics of passive fluids in the presence of an external potential while imposing all the thermodynamic relations one finds in the passive fluid. The active nature of the fluid does not need to modify the constitutive relations, in particular their thermodynamic character, to be compatible with observations. As we shall see in the next sections, this is further verified by considering the scaling behavior of the transport terms under the dynamical renormalization group.

IV. DYNAMICAL RENORMALIZATION GROUP

Having derived the Toner-Tu equations from first principles, we use the fact that several of our transport coefficients

are pure functions of thermodynamic parameters, (6), to our advantage; namely, we constrain the scaling behavior of the transport coefficients. Using standard notations, we rescale fields and coordinates as $(\vec{v}_\perp, \delta n, \vec{x}_\perp, x_\parallel, t) = (b^\chi \vec{v}'_\perp, b^\chi \delta n', b^\zeta \vec{x}'_\perp, b^\zeta x'_\parallel, b^z t')$. With this choice of scalings the equations of motion take the same form as before, but with rescaled transport coefficients given by

$$(w_1, g_{3,4,5})' = b^{z-1+\chi}[(w_1, g_{3,4,5}) + \text{graphs}], \quad (15a)$$

$$(w_{2,3}, g_{1,2})' = b^{z-\zeta+\chi}[(w_{2,3}, g_{1,2}) + \text{graphs}], \quad (15b)$$

$$(\xi_n, D_v)' = b^{z-1-\zeta}[(\xi_n, D_v) + \text{graphs}], \quad (15c)$$

$$(\xi_\parallel, D_\parallel)' = b^{z-2\zeta}[(\xi_\parallel, D_\parallel) + \text{graphs}], \quad (15d)$$

$$(\xi_{v,\perp}, D_\perp)' = b^{z-2}[(\xi_{v,\perp}, D_\perp) + \text{graphs}] \quad (15e)$$

$$\Delta' = b^{z-\zeta+1-d-2\chi}(\Delta + \text{graphs}), \quad (15f)$$

where “graphs” represents perturbative corrections that are obtained from integrating out the fast modes with wave vector $b^{-1}\Lambda \leq |\vec{q}_\perp| \leq \Lambda$ and Λ is a UV cutoff [33,34]. As was shown in [3,34], the asymptotic $\vec{q} \rightarrow 0$ behavior of the correlators is determined by the fixed-point values of the scaling exponents ζ , z , and χ . Therefore, a lot of effort has been devoted to identifying the critical points of these β functions using both clever symmetry arguments [3,35–37] or through more direct computation [38,39].

The simplest such fixed-point example is the linear one, where nonlinear terms are ignored. It is given by $(z, \zeta, \chi) = (2, 1, \frac{2-d}{2})$. However, the linear fixed point is unstable for $d < 4$, as nonlinearities become relevant in the IR. Another example is found in [3], where Toner and Tu obtained exact exponents in $d = 2$ taking advantage of an emergent pseudo-Galilean symmetry. They obtain $(z, \zeta, \chi) = (6/5, 3/5, -1/5)$. These results, however, are invalidated by the more accurate analysis of [5], which agrees with our expressions (13a) and (13b), due to the presence of new nonlinearities ignored in the original work.

A. Entropy conservation and nonlinear terms

We now argue that we can fix the values of the critical exponents in (15) exactly if we assume, as has been demonstrated numerically [40], an exact scaling for the noise correlator coefficient Δ . Looking at the nonlinear terms in our hydrodynamic expressions, the associated bare transport coefficients g_i and w_i all depend on global equilibrium thermodynamic properties (like ρ_0 , n_0 , v_0 , and P), their thermodynamic derivatives (such as susceptibilities, compressibilities, ...) and derivatives of the external potential $(\frac{\partial U}{\partial n})_{v^2}$ and $(\frac{\partial U}{\partial v^2})_n$. As these quantities are thermodynamic in origin, they must not receive perturbative corrections as there are no hydrodynamic modes in global equilibrium, where these quantities are defined, that can renormalize these values.

More precisely, the hydrodynamic constitutive relations are not just the most general expressions compatible with symmetries; they are constrained by the second law of thermodynamics. For example, at first order in derivatives, the second law imposes well-known constraints on the signs of the dissipative transport coefficients. Less appreciated is the fact that for ideal fluids the second law forces particular ther-

modynamic relations to be satisfied. To demonstrate this for the passive fluid, consider the entropy of the system, which is a function of the thermodynamic variables $s = s(n, \vec{v}^2)$, and its associated entropy flux $\vec{J}_s = (s - P/T_0)\vec{v}$ with T_0 some constant temperature. Then, using the thermodynamic relation $T_0 ds = -\mu dn - \vec{v} \cdot d\vec{g}$, it can be shown that the ideal fluid Eqs. (1) imply entropy is conserved, $\partial_t s + \vec{\nabla} \cdot \vec{J}_s = 0$. This result holds generically in hydrodynamics and tells us that the ideal fluid does not produce entropy as a consequence of local equilibrium.

In the presence of an external potential U , the entropy of the ideal fluid satisfies a modified conservation law

$$T_0(\partial_t s + \vec{\nabla} \cdot \vec{J}_s) + U \vec{v}^2 = 0. \quad (16)$$

We emphasize that this equation is a consequence of the ideal-fluid equations of motion and the definition of the entropy s . That (16) holds identically for all solutions of the ideal-fluid equations of motion is subsequently a consequence of thermodynamic relations between coefficients appearing in the other transport equations. As such, (16) holds for any solution of the fluid equations of motion including those in which the longitudinal fluctuations are integrated out using (11). This can be seen directly if we expand (16) up to order two in fluctuations and eliminate δv_\parallel using (11) to obtain

$$\begin{aligned} T_0 \partial_t \delta s + \varphi \vec{\nabla}_\perp \cdot \vec{v}_\perp + \pi \partial_\parallel \delta n + f_1 \delta n (\vec{\nabla}_\perp \cdot \vec{v}_\perp) \\ + f_2 (\vec{v}_\perp \cdot \vec{\nabla}_\perp) \delta n + f_3 \partial_\parallel |\vec{v}_\perp|^2 + f_4 \partial_\parallel (\delta n)^2 + O(\partial^2) = 0, \end{aligned} \quad (17)$$

where φ , π , and the f_i are again functions that depend on the thermodynamics and the external potential (see Appendix B). Applying the definition of s in terms of the other charges,

$$T_0 \partial_t \delta s = -\mu \partial_t \delta n - (v_0 + \delta v_\parallel) \delta_t g_\parallel - \vec{v}_\perp \cdot \partial_t \vec{g}_\perp, \quad (18)$$

employing the reduced equations of motion (13a) and (13b), and also the reduced equation for $\partial_t g_\parallel$, we indeed find that (17) is identically satisfied.²

For the bare theory at the ideal level, satisfaction of (16) independently of the flow is just a consequence of thermodynamic relations between transport coefficients. Once we proceed with the first step of the dynamical renormalization group and integrate out the fast modes we will find that Eqs. (13a), (13b), and (17) have the same form, but for renormalized values of the transport coefficients. Our argument is that the renormalized dynamical Eqs. (13a) and (13b) should still identically solve (17) upon using thermodynamic relations.

²One needs to be careful as in (13b) the time derivative acts on velocity fluctuations, while to check (17) we need the time derivative to act on the momentum.

To clarify this last step, we can look at a specific example. Inserting the equations of motion in (17) we find many terms, some of the simplest ones being

$$\left(f_3 - \mu_0 w_3 - \frac{1}{2} v_0 \rho_0\right) \partial_{\parallel} |\vec{v}_{\perp}|^2 = 0, \quad (19a)$$

$$\left(\frac{\partial \rho}{\partial n} v_0^2 - 2 \frac{\partial \rho}{\partial v^2} v_0^3 \alpha_1 + \kappa - v_0 \alpha_1 \rho_0 + f_2 + \mu_0 w_1\right) \vec{v}_{\perp} \cdot \vec{\nabla} \delta n = 0, \quad (19b)$$

which, as expected, vanish for the bare values of the transport coefficients reported in Appendix B. Suppose that we perform the first step of the dynamical renormalization group on Eqs. (13a), (13b), and (17). This in general will leave the form of the equations invariant, but will change the values of the transport coefficients due to graphical corrections, e.g., $f_i \rightarrow f'_i = f_i + \delta f_i$. This process will not affect the coefficients in the longitudinal momentum equation, since the fluctuation δv_{\parallel} is not dynamical. Subsequently, we find the same constraints as in (19), but written in terms of the renormalized values f'_i and w'_i . Finally, imposing that the equation is still identically satisfied for the new values of the coefficients amounts to requiring that the perturbative corrections obey

$$\delta f_2 + \mu_0 \delta w_1 = 0, \quad \delta f_3 - \mu_0 \delta w_3 = 0. \quad (20)$$

Because the chemical potential μ_0 is a nonuniversal function, and because it does not appear in our equations of motion, the only way to make the above expressions vanish is to have that $\delta f_2 = \delta f_3 = \delta w_1 = \delta w_3 = 0$. At order two in fluctuations we find one last nontrivial relation $\delta f_4 - \mu_0 \delta w_2 = 0$, which implies that w_2 cannot receive perturbative corrections either. Therefore, the nonlinear transport coefficients w_1 , w_2 , and w_3 in (13a) cannot receive graphical corrections if the ideal fluid is not to produce entropy.

To find all the possible constraints one should expand the entropy Eq. (16) to order three in fluctuations, although the resulting expressions become rather long. Specifically, at order three in fluctuations we find the constraints

$$\delta g_i + \delta l_i = 0, \quad (21)$$

where l_i are coefficients that arise expanding the entropy Eq. (16) up to order three in fluctuations. These relations seem less restricting than those in (20); however, had we used n and \vec{g} as our variables of choice to write (13b) (which are the natural variables to study entropy conservation, instead of n and \vec{v}), the above constraints would present thermodynamic derivatives $\partial \vec{v} / \partial \vec{g}$ in front of δg_i , suggesting again that the perturbative corrections of g_i and l_i must be independent, and thus they should all vanish.

The usual arguments used to forbid graphical renormalization are grounded in the continuous symmetries of the equations of motion (e.g., the Galilean symmetry in Navier-Stokes hydrodynamics [34]). Here we rely on another continuous symmetry, entropy (non-)conservation. This latter symmetry can be formalized in terms of Kubo Martin Schwinger (KMS) symmetries of the Schwinger-Keldysh path integral (see [26] for a discussion). This argument, perhaps unknowingly, is what prompted Toner and Tu to assert that λ_n in (2b) should be kept fixed at 1 [3]. Here, we propose that the

same argument can be generalized to include all the ideal-fluid transport coefficients.

We remark that this conclusion is in agreement with all known exact results obtained in the past based on the pseudo-Galilean invariance. Specifically, the results for Malthusian flocks in $d = 2$ [35,41] and $d = 3$ [38,42] and incompressible flocks in $d = 3$ [43] all agree that nonlinear terms should not receive perturbative corrections. The only exception is incompressible polar fluids in the disordered phase, for which perturbative corrections have been explicitly computed [44]. Nevertheless, even this last case can be understood from our argument. In the incompressible disordered phase, both chemical potentials μ_0 and v_0 are zero, moreover δv_{\parallel} cannot be integrated out. Hence the thermodynamics of the system simplifies significantly and we find $\delta l_i + \delta g_i = 0$ without any thermodynamic quantity in front of δg_i , thus the only constraint is that perturbative corrections to the entropy equation transport coefficients must be equal to perturbative corrections to the momentum equation ones.

B. Exact scaling exponents

Having understood that the nonlinear terms should not receive perturbative corrections, we can impose this on the scalings of the coefficients g_i and w_i in (13a) and (13b), finding two equations for the fixed point:

$$z - 1 + \chi = 0, \quad z - \zeta + \chi = 0. \quad (22)$$

To completely fix the exponents we need a third equation and as such we will employ the hyperscaling relation. This additional constraint follows from assuming that the scaling of Δ is exact. The reason for this is very practical, and comes from the fact that recent simulations suggest it is a universal law [40]. Furthermore, a constant noise correlator Δ should in general receive contributions from at least two nonlinearities, each carrying a power of \vec{q} . Corrections are then $O(q^2)$, which are irrelevant compared to Δ in the hydrodynamic limit. Imposing this last condition

$$z - \zeta + 1 - d - 2\chi = 0 \quad (23)$$

we find the exact scaling exponents

$$z = \frac{2+d}{3}, \quad \zeta = 1, \quad \chi = \frac{1-d}{3} \quad (24)$$

valid for all $2 \leq d \leq 4$, assuming again exactness of the scaling of Δ . These expressions correctly interpolate to the mean-field theory results which hold for $d \geq 4$. They are in perfect agreement with recent state-of-the-art numerics on the Vicsek model [40] which found discrepancies when

TABLE I. Critical exponents for Toner and Tu [3] and Vicsek [40], compared to the exact exponent computed in this work.

	$d = 2$			$d = 3$		
	TT98	Vicsek	This work	TT98	Vicsek	This work
χ	-1/5	-0.31(2)	-1/3	-3/5	-0.62	-2/3
ζ	3/5	0.95(2)	1	4/5	1	1
z	6/5	1.33(2)	4/3	8/5	1.77	5/3
α	8/5	1.67(2)	5/3	14/9	1.59(3)	23/15

comparing to the standard Toner and Tu expressions. These are outlined in Table I. We also included in the table the large number fluctuation exponents $\langle \delta N \rangle^2 \sim \langle N \rangle^\alpha$. These exponents should be 1 for thermal phases, while polar active matter hydrodynamics predicts $\alpha = 1 + (d + \zeta + 2\chi - 1)/d$ [1,40,45]. In the specific case of $d = 2$ our argument also matches recent studies based on symmetry arguments [46].

As expected, $\chi < 0$ in $d = 2$, implying that velocity fluctuations are weak and do not destroy the ordered phase. The Mermin-Wagner theorem would forbid continuous phase transitions below two dimensions; however, active matter systems defy this expectation due to their “nonequilibrium nature.” In the past this has been attributed to λ_i terms in (2). On the contrary, in our approach, the λ_i are constrained by local equilibrium and do not meaningfully depend on any activity. Yet nevertheless, as we conclude this section, we find we have reproduced the necessary scaling properties of the transport coefficients. This adds to our evidence from previous sections where we recovered the transport phenomenology of the active fluid.

V. OUTLOOK

We have demonstrated that polar active fluids correspond to passive fluids lacking boost symmetry when the latter are supplemented with a nonthermal noise term and a potential inducing spontaneous symmetry breaking. In particular, thermodynamic relations must continue to hold in the ordered phase. Consequently, assuming exactness of the noise correlator following standard arguments [40], we precisely derived expressions for the critical exponent. These expressions align excellently with numerical simulations of the Vicsek model and also correlate well with experimental observations

on epithelial cells [47], self-propelled rollers [28], bacterial colonies [48], polar rods [49], and earlier simulations of the Vicsek model [50,51]. This paves the way for more precise computations, where the constraint derived in our approach can give better agreement with experimental and numerical results. It would also be interesting to derive the constraints we have found from a kinetic theory using the approaches described in [29,30] and test them on more exotic types of matter [52–54].

We expect our argument to be valid for generic systems, like Malthusian flocks [35,46] or incompressible phases [2]. It would also be interesting to show precisely how entropy conservation constrains the ideal-fluid transport coefficients at the level of the Feynman diagrams, even just for simple fluids without external potential U . Moreover, while we omitted the consideration of temperature, as is customary in the Toner-Tu model, it is straightforward to introduce an additional scalar quantity, T , into our formalism. This leads to a conservation equation associated with time translation. Naturally, we anticipate this equation to undergo relaxation similar to that of the equation for \bar{g} [26]. Understanding the implications of this relaxed equation could provide insight into the nature of entropy conservation/production within the effective active matter systems.

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APPENDIX A: CONSTITUTIVE RELATIONS FOR ORDER-ONE BOOST-AGNOSTIC HYDRODYNAMICS

In this appendix we give the constitutive relations for a boost-agnostic fluid in d spatial dimensions up to and including first order in derivatives. The constitutive relations for the currents j^i and τ^{ij} are [20]

$$j^i = nv^i - \left(\gamma_{00} P^{ij} \partial_j \mu + \zeta_{00} \frac{v^i v^j}{|v|^2} \partial_j \mu + \gamma_{01} \frac{P^{i(j} v^{k)}}{|v|} \sigma_{jk} + \zeta_{01} \frac{v^i v^j v^k}{2|v|^3} \sigma_{jk} + \frac{\zeta_{02}}{2} \frac{v^i}{|v|} P^{jk} \sigma_{jk} \right), \quad (\text{A1a})$$

$$\begin{aligned} \tau^{ij} = & \rho v^i v^j + P \delta^{ij} - \left[\eta \left(P^{k(i} P^{j)l} - \frac{1}{d-1} P^{ij} P^{kl} \right) \sigma_{kl} + 2\gamma_{01} \frac{P^{k(i} v^{j)}}{|v|} \partial_k \mu + \zeta_{01} \frac{v^i v^j v^k}{|v|^3} \partial_k \mu \right. \\ & + \zeta_{02} \frac{v^k}{|v|} P^{ij} \partial_k \mu + 2\gamma_{11} \frac{v^{(i} P^{j)(k} v^{l)}}{|v|^2} \sigma_{kl} + \frac{1}{2} \zeta_{11} \frac{v^i v^j v^k v^l}{|v|^4} \sigma_{kl} + \frac{\zeta_{12}}{2} \left(P^{ij} \frac{v^k v^l}{|v|^2} + \frac{v^i v^j}{|v|^2} P^{kl} \right) \sigma_{kl} \\ & \left. + \frac{\zeta_{22}}{2} P^{ij} P^{kl} \sigma_{kl} \right], \end{aligned} \quad (\text{A1b})$$

where $\sigma_{ij} = \partial_i v_j + \partial_j v_i$, $P^{ij} = \delta^{ij} - \frac{v^i v^j}{|v|^2}$ is the projector orthogonal to the velocity, and μ is the chemical potential. The momentum constitutive relation remains fixed at its ideal-fluid form, $\bar{g} = \rho \bar{v}$. Positivity of entropy production constrains the

matrices of transport coefficients

$$\begin{pmatrix} \zeta_{00} & \zeta_{01} & \zeta_{02} \\ \zeta_{01} & \zeta_{11} & \zeta_{12} \\ \zeta_{02} & \zeta_{12} & \zeta_{22} \end{pmatrix} \geq 0, \quad \begin{pmatrix} \gamma_{00} & \gamma_{01} \\ \gamma_{01} & \gamma_{11} \end{pmatrix} \geq 0, \quad \eta \geq 0, \quad (\text{A2})$$

i.e., the ζ and γ matrices are positive semidefinite. We can rewrite the constitutive relations using n instead of μ as a thermodynamic variable, which is more natural if we want to compare our results with [3]

$$j^i = nv^i - \left(\bar{\gamma}_{00} P^{ij} \partial_j n + \bar{\zeta}_{00} \frac{v^i v^j}{|v|^2} \partial_j n + \bar{\gamma}_{01} \frac{P^{i(j} v^{k)} \sigma_{jk}}{|v|} + \bar{\zeta}_{01} \frac{v^i v^j v^k}{2|v|^3} \sigma_{jk} + \frac{\bar{\zeta}_{02}}{2} \frac{v^i}{|v|} P^{jk} \sigma_{jk} + m P^{ij} \partial_j v^2 \right), \quad (\text{A3a})$$

$$\begin{aligned} \tau^{ij} = & \rho v^i v^j + P \delta^{ij} - \left[\eta \left(P^{k(i} P^{j)l} - \frac{1}{d-1} P^{ij} P^{kl} \right) \sigma_{kl} + 2 \bar{\gamma}_{01} \frac{P^{k(i} v^{j)} \partial_k n}{|v|} + \bar{\zeta}_{01} \frac{v^i v^j v^k}{|v|^3} \partial_k n \right. \\ & + \bar{\zeta}_{02} \frac{v^k}{|v|} P^{ij} \partial_k n + 2 \bar{\gamma}_{11} \frac{v^{(i} P^{j)(k} v^{l)} \sigma_{kl}}{|v|^2} + \frac{1}{2} \bar{\zeta}_{11} \frac{v^i v^j v^k v^l}{|v|^4} \sigma_{kl} + \frac{\bar{\zeta}_{12}}{2} \left(P^{ij} \frac{v^k v^l}{|v|^2} + \frac{v^i v^j}{|v|^2} P^{kl} \right) \sigma_{kl} \\ & \left. + \frac{\bar{\zeta}_{22}}{2} P^{ij} P^{kl} \sigma_{kl} + 2 \frac{t}{|v|} v^{(i} P^{j)k} \partial_k v^2 + \frac{n}{|v|} P^{ij} v^k \partial_k v^2 \right], \end{aligned} \quad (\text{A3b})$$

where we redefined the transport coefficients by absorbing the susceptibilities

$$\bar{\gamma}_{00} = \gamma_{00} \left(\frac{\partial \mu}{\partial n} \right)_{v^2}, \quad \bar{\gamma}_{01} = \gamma_{01}, \quad \bar{\zeta}_{00} = \zeta_{00} \left(\frac{\partial \mu}{\partial n} \right)_{v^2}, \quad (\text{A4a})$$

$$\bar{\zeta}_{01} = \zeta_{01} + 2 \zeta_{00} |\bar{v}| \left(\frac{\partial \mu}{\partial v^2} \right)_n, \quad \bar{\zeta}_{02} = \zeta_{02}, \quad m = \gamma_{00} \left(\frac{\partial \mu}{\partial v^2} \right)_n, \quad (\text{A4b})$$

$$\bar{\gamma}_{01} = \gamma_{01} \left(\frac{\partial \mu}{\partial n} \right)_{v^2}, \quad \bar{\gamma}_{11} = \gamma_{11}, \quad \bar{\zeta}_{01} = \zeta_{01} \left(\frac{\partial \mu}{\partial n} \right)_{v^2}, \quad (\text{A4c})$$

$$\bar{\zeta}_{11} = \zeta_{11} + 2 \zeta_{01} |\bar{v}| \left(\frac{\partial \mu}{\partial v^2} \right)_n, \quad \bar{\zeta}_{02} = \zeta_{02} \left(\frac{\partial \mu}{\partial n} \right)_{v^2}, \quad \bar{\zeta}_{12} = \zeta_{12}, \quad (\text{A4d})$$

$$\bar{\zeta}_{22} = \zeta_{22}, \quad t = \gamma_{01} \left(\frac{\partial \mu}{\partial v^2} \right)_n, \quad n = \zeta_{02} \left(\frac{\partial \mu}{\partial v^2} \right)_n. \quad (\text{A4e})$$

In spite of the fact that there are ten independent order-one transport coefficients, the equations of motion linearized around a constant background velocity $\bar{v} = v_0 \hat{x}_{\parallel}$ will be such that we can absorb all of them into three order-one diffusive coefficients in the charge conservation equation and four transport coefficients in the momentum equations.

APPENDIX B: TRANSPORT COEFFICIENTS FROM INTEGRATING OUT THE LONGITUDINAL MODE

We introduce a notation, also used in the main text, for the thermodynamic derivatives of the external potential U , the pressure P , the kinetic mass density ρ , and the entropy density s :

$$\begin{aligned} \left(\frac{\partial F}{\partial n} \right)_{v^2} &= F_n, & \left(\frac{\partial F}{\partial \bar{v}^2} \right)_n &= F_v, & \left(\frac{\partial^2 F}{\partial n^2} \right)_{v^2} &= F_{n^2}, \\ \left(\frac{\partial^2 F}{\partial (\bar{v}^2)^2} \right)_n &= F_{v^2}, & \frac{\partial^2 F}{\partial n \partial \bar{v}^2} &= F_{n,v} \end{aligned} \quad (\text{B1})$$

with $F = \{U, P, \rho, s\}$. These thermodynamic derivatives are evaluated on the background, at $n = n_0$ and $\bar{v}^2 = v_0^2$. We remark that, without loss of generality, we set $U_{v^2} = U_{n,v} = 0$. Reinstating these derivatives simply changes the values of α_i in (11), but does not affect any conclusion.

In the main text, when we define the effective equation for fluctuations of the charge density once the longitudinal mode is integrated out, we make use of the following definitions:

$$v_2 = v_0 - \alpha_1 n_0, \quad w_1 = 1, \quad (\text{B2a})$$

$$w_2 = \alpha_1 + \alpha_2 n_0, \quad w_3 = \frac{n_0}{2v_0}, \quad (\text{B2b})$$

$$\xi_n = \alpha_3 n_0 + \bar{\gamma}_{01} + \bar{\zeta}_{02}, \quad D_{\parallel} = n_0 \alpha_4 + \bar{\zeta}_{00} - \alpha_1 \bar{\zeta}_{01}, \quad (\text{B2c})$$

$$D_{\perp} = \bar{\gamma}_{00} - \alpha_1 (2v_0 m + \bar{\gamma}_{01}). \quad (\text{B2d})$$

Similarly, for the transverse velocity fluctuations we find

$$\kappa = \frac{1}{\rho_0} (P_n - 2v_0 \alpha_1 P_v), \quad g_1 = -\alpha_1, \quad (\text{B3a})$$

$$g_2 = \frac{1}{\rho_0} (n_0 \alpha_1 \rho_n - 2\alpha_1^2 n_0 v_0 \rho_v + 2v_0 \alpha_4 U_v) - \alpha_1, \quad (\text{B3b})$$

$$\begin{aligned} g_3 = & \frac{P_{n^2}}{2\rho_0} + \frac{P_v}{\rho_0} (\alpha_1^2 - 2v_0 \alpha_2) + \frac{\kappa (2v_0 \alpha_1 \rho_v - \rho_n)}{2\rho_0} \\ & - \frac{2v_0 \alpha_1 P_{n,v}}{\rho_0} + \frac{2v_0^2 \alpha_1^2 P_{v^2}}{\rho_0}, \end{aligned} \quad (\text{B3c})$$

$$g_4 = 1 - \frac{1}{\rho_0}(n_0\rho_n - 2v_0n_0\alpha_1\rho_v - 2v_0\alpha_3U_v), \quad g_5 = 1, \quad (B3d)$$

$$D_v = \frac{1}{\rho_0}(2v_0\alpha_4P_v + \tilde{\gamma}_{01} + \tilde{\xi}_{02} - \alpha_1[2v_0(t+n) + \tilde{\gamma}_{11} + \tilde{\xi}_{12}]),$$

$$\xi_v = \frac{1}{\rho_0}(2v_0\alpha_3P_v + \tilde{\xi}_{22}), \quad (B3e)$$

$$\xi_{\parallel} = \frac{\tilde{\gamma}_{11}}{\rho_0}, \quad \xi_{\perp} = \frac{\eta}{\rho_0}. \quad (B3f)$$

Finally, the coefficients that appear in the entropy Eq. (17) are

$$\varphi = s_0T_0 - P_0 + 2\alpha_3v_0^3U_v, \quad (B4a)$$

$$\pi = v_0(T_0s_n - P_n) + \alpha_1(P_0 - s_0T_0 + 2v_0^2(P_v - T_0s_v)) + 2\alpha_4v_0^3U_v, \quad (B4b)$$

$$f_1 = T_0s_n - P_n + v_0(2\alpha_1P_v - 2\alpha_1T_0s_v - 3\alpha_3U_n), \quad (B4c)$$

$$f_2 = T_0s_n - P_n + 2v_0\alpha_1(P_v - T_0s_v), \quad (B4d)$$

$$f_3 = \frac{P_0 - s_0T_0}{2v_0}, \quad (B4e)$$

$$f_4 = \frac{v_0(T_0s_n^2 - P_n^2)}{2} + \alpha_1(P_n - T_0s_n) + 2v_0^2\alpha_1(P_{n,v} - T_0s_{n,v})$$

$$- 3\alpha_1^2v_0(P_v - T_0s_v) - 2v_0^3\alpha_1^2(P_{v^2} - T_0s_{v^2}) + \alpha_2(P_0 - T_0s_0) + 2v_0\alpha_2(P_v - T_0s_v) - \frac{3}{2}\alpha_4v_0U_n. \quad (B4f)$$

APPENDIX C: DETAILS OF THE ENTROPY ARGUMENT

In this appendix we add extra details to the argument that allows us to fix the exact scalings of nonlinear terms.

We remind the reader once more that the hydrodynamic equations are not just the most general expressions compatible with symmetries, but are also constrained by the second law of thermodynamics. In particular, we always require that the ideal fluid (i.e., lowest order in derivatives) conserves entropy, while higher derivative dissipative corrections produce entropy. Our argument to fix the scaling of the nonlinear terms under a dynamical renormalization group (DRG) is grounded on the idea that entropy conservation of the ideal fluid should be conserved along the RG flow.

We are interested in the ideal fluid, hence we will consider the equations of hydrodynamics up to order one in derivatives. First, focus on the isotropic theory, namely, Eqs. (2) truncated at order one, where the values of the λ_i parameters are given in (6):

$$\partial_t \vec{v} + \lambda_1(\vec{v} \cdot \vec{\nabla})\vec{v} + \lambda_2(\vec{\nabla} \cdot \vec{v})\vec{v} + \lambda_3\vec{\nabla}|\vec{v}|^2 + \vec{\nabla}P_1 + \vec{v}(\vec{v} \cdot \vec{\nabla}P_2) = U\vec{v} + O(\partial^2), \quad (C1a)$$

$$\partial_t n + \lambda_n\vec{\nabla} \cdot (n\vec{v}) + O(\partial^2) = 0, \quad (C1b)$$

where we also set to zero the noise term, which is not important for this derivation.

Consider now the entropy density $s = s(n, \vec{g})$ given as a function of the thermodynamic variables n and \vec{g} . The entropy flux, at lowest order in derivatives, takes the form $\vec{J}_s = (s - P/T_0)\vec{v}$, where T_0 is some constant temperature. At lowest order in derivatives entropy is conserved, up to the external field U , so that we have the entropy conservation law (16)

$$\partial_t s + \vec{\nabla} \cdot \vec{J}_s + U\vec{v}^2 = 0. \quad (C2)$$

To show this we can use the thermodynamic relation

$$T_0 ds = -\mu dn - \vec{v} \cdot d\vec{g}, \quad (C3)$$

to substitute the time derivative of the entropy $\partial_t s$ in terms of the time derivatives of $\partial_t n$ and $\partial_t \vec{g}$, where we remind the reader that $\vec{g} = \rho(n, \vec{v}^2)\vec{v}$ in terms of the velocity and $\mu = \mu(n, \vec{v})$ is the chemical potential. Using now the equations of motion at lowest order in derivatives (C1) we find that (C2) is identically satisfied by the dynamics.

This means that the full nonlinear isotropic theory identically conserves entropy (C2) (at lowest order in derivatives). Clearly, Eq. (C2) must hold even when we expand in fluctuations and integrate out the heavy mode δv_{\parallel} . We can show this explicitly: consider the anisotropic equations of motion (13a) and (13b), as always truncated at lowest order in derivatives

$$\partial_t \delta n + v_2 \partial_{\parallel} \delta n + n_0 \vec{\nabla}_{\perp} \cdot \vec{v}_{\perp} + w_1 \vec{\nabla}_{\perp}(\vec{v}_{\perp} \delta n) - w_2 \partial_{\parallel}(\delta n)^2 - w_3 \partial_{\parallel}|\vec{v}_{\perp}|^2 = 0, \quad (C4a)$$

$$\partial_t \vec{v}_{\perp} + v_0 \partial_{\parallel} \vec{v}_{\perp} + \kappa \vec{\nabla}_{\perp} \delta n + g_1 \delta n \partial_{\parallel} \vec{v}_{\perp} + g_2 \vec{v}_{\perp} \partial_{\parallel} \delta n + g_3 \vec{\nabla}_{\perp}(\delta n)^2 + g_4 \vec{v}_{\perp}(\vec{\nabla}_{\perp} \cdot \vec{v}_{\perp}) + g_5(\vec{v}_{\perp} \cdot \vec{\nabla}_{\perp})\vec{v}_{\perp} = 0. \quad (C4b)$$

We can apply the same procedure used to obtain the above equations also to the entropy equation. Expanding (C2) up to order three in fluctuations³ and plugging in the expression for the heavy mode (11) we obtain (17):

$$T_0 \partial_t \delta s + \varphi \vec{\nabla}_{\perp} \cdot \vec{v}_{\perp} + \pi \partial_{\parallel} \delta n + f_1 \delta n(\vec{\nabla}_{\perp} \cdot \vec{v}_{\perp}) + f_2(\vec{v}_{\perp} \cdot \vec{\nabla}_{\perp})\delta n + f_3 \partial_{\parallel}|\vec{v}_{\perp}|^2 + f_4 \partial_{\parallel}(\delta n)^2 + l_1 \partial_{\parallel}(\delta n)^3 + l_2|\vec{v}_{\perp}|^2 \partial_{\parallel} \delta n + l_3 \delta n^2 \vec{\nabla}_{\perp} \cdot \vec{v}_{\perp} + l_4|\vec{v}_{\perp}|^2 \vec{\nabla}_{\perp} \cdot \vec{v}_{\perp} + l_5 \vec{v}_{\perp} \cdot \vec{\nabla}_{\perp}(\delta n)^2 + l_6 \delta n \partial_{\parallel}|\vec{v}_{\perp}|^2 + l_7(\vec{v}_{\perp} \cdot \vec{\nabla}_{\perp})|\vec{v}_{\perp}|^2 = 0. \quad (C5)$$

The values of f_i are found in Appendix B, while the l_i are extremely long expressions composed of several thermodynamic derivatives which can be obtained in the same manner, but which we avoid writing here.

³Order three is necessary to constrain all the coefficients w_i which appear in the velocity Eq. (C4).

Finally, we can use again (C3) and (11) to substitute $\partial_t \delta s$ in the above equation in terms of time derivatives of n and \vec{v}_\perp . Upon also using the equations of motion (C4) we find

$$\begin{aligned} & \left(\varphi + \mu_0 n_0 + 2v_0^3 \alpha_3 \frac{\partial U}{\partial v^2} + \rho_0 v_0^2 \right) (\vec{\nabla}_\perp \cdot \vec{v}_\perp) + \left(\pi + \frac{\partial P}{\partial n} v_0 + \frac{\partial \rho}{\partial n} v_0^3 - 2 \frac{\partial P}{\partial v^2} v_0^2 \alpha_1 - 2 \frac{\partial \rho}{\partial v^2} v_0^4 \alpha_1 + 2 \frac{\partial U}{\partial v^2} v_0^3 \alpha_4 \right. \\ & \quad \left. + \mu_0 v_2 - 2v_0^2 \alpha_1 \rho_0 \right) \partial_\parallel \delta n + \left(\frac{\partial \rho}{\partial n} v_0^2 - 2 \frac{\partial \rho}{\partial v^2} v_0^3 \alpha_1 - 6 \frac{\partial U}{\partial v^2} v_0^2 \alpha_1 \alpha_3 + \frac{\partial \mu}{\partial n} n_0 - 2 \frac{\partial \mu}{\partial v^2} v_0 n_0 \alpha_1 - 2 v_0 \alpha_1 \rho_0 + f_1 + \mu_0 w_1 \right) \\ & \quad \times \delta n (\vec{\nabla}_\perp \cdot \vec{v}_\perp) + \left(\frac{\partial \rho}{\partial n} v_0^2 - 2 \frac{\partial \rho}{\partial v^2} v_0^3 \alpha_1 + \kappa - v_0 \alpha_1 \rho_0 + f_2 + \mu_0 w_1 \right) (\vec{v}_\perp \cdot \vec{\nabla}_\perp) \delta n + (2f_3 - v_0 \rho_0 - 2\mu_0 w_3) \frac{1}{2} \partial_\parallel |\vec{v}_\perp|^2 \\ & \quad + \left(\frac{\partial^2 P}{\partial n^2} v_0 + \frac{\partial^2 \rho}{\partial n^2} v_0^3 + \frac{\partial \mu}{\partial n} v_2 - \frac{\partial P}{\partial n} \alpha_1 - 4 \frac{\partial^2 P}{\partial n \partial v^2} v_0^2 \alpha_1 - 5 \frac{\partial \rho}{\partial n} v_0^2 \alpha_1 - 4 \frac{\partial^2 \rho}{\partial n \partial v^2} v_0^4 \alpha_1 - 2 \frac{\partial \mu}{\partial v^2} v_0 v_2 \alpha_1 + 4 \frac{\partial P}{\partial v^2} v_0 \alpha_1^2 \right. \\ & \quad \left. + 4 \frac{\partial^2 P}{\partial (v^2)^2} v_0^3 \alpha_1^2 + 12 \frac{\partial \rho}{\partial v^2} v_0^3 \alpha_1^2 + 4 \frac{\partial^2 \rho}{\partial (v^2)^2} v_0^5 \alpha_1^2 - 4 \frac{\partial P}{\partial v^2} v_0^2 \alpha_2 - 4 \frac{\partial \rho}{\partial v^2} v_0^4 \alpha_2 - 6 \frac{\partial U}{\partial v^2} v_0^2 \alpha_1 \alpha_4 + 4 v_0 \alpha_1^2 \rho_0 - 4 v_0^2 \alpha_2 \rho_0 \right. \\ & \quad \left. + 2f_4 - 2\mu_0 w_2 \right) \frac{1}{2} \partial_\parallel (\delta n)^2 + \dots = 0, \end{aligned} \quad (C6)$$

where the dots represent terms that are order three in fluctuations and which we avoid writing to not clutter the expressions even more. As expected, we find that Eq. (C5) is identically satisfied by the equations of motion (C4). This is in agreement with the statement of the main text: the values of the ideal-fluid transport coefficients [e.g., the w_i and g_i in (C4)] are not arbitrary, but must take specific values for the entropy current to satisfy (C5) identically on shell.

Thus, we have understood that in the bare theory Eq. (C5) is identically obeyed by the dynamics. Despite not being an independent degree of freedom, ideal-fluid entropy conservation is a symmetry of the system, which must be preserved along the RG flow when we integrate out the fast modes. Like all symmetries, it constrains the perturbative renormalization of the parameters, specifically it forbids graphical corrections to the ideal-fluid transport coefficients, as we now explain.

Given the equations of motion (C4) (to which we add back the dissipative corrections and the noise term) we proceed with the first step of the DRG [3,34]. Namely, we separate the modes \vec{v}_\perp and δn in Fourier space in fast modes, for which $b^{-1}\Lambda \leq |\vec{q}_\perp| \leq \Lambda$, and slow modes, given by $|\vec{q}_\perp| \leq b^{-1}\Lambda$, for some cutoff Λ . Given the equations are nonlinear, there will be mixing between the modes, hence after we integrate out the fast modes (as a power series in the nonlinearities and by averaging over the fast noise modes) we obtain a set of equations identical to our bare theory (C4), but with modified or dressed transport coefficients⁴

$$w'_i = w_i + \delta w_i, \quad g'_i = g_i + \delta g_i. \quad (C7)$$

Notice, in particular, that the entropy Eq. (C5), despite not being part of the dynamics, also receives graphical corrections. Indeed, when we integrate out the fast modes we can check the effect it has also on (C5). This suggests that also f_i and l_i

get modified as

$$f'_i = f_i + \delta f_i, \quad l'_i = l_i + \delta l_i. \quad (C8)$$

The terms δw_i , δg_i , etc., are what we refer to as perturbative or graphical corrections, since they arise as a power series in the nonlinearities which can be computed (at least in principle) using diagrams.

We can now study the effect that such perturbative corrections would have on the entropy conservation law. Consider then the renormalized Eq. (C5), written in terms of l'_i and f'_i , and substitute one last time $\partial_t \delta s$ in favor of $\partial_t \delta n$ and $\partial_t \vec{v}_\perp$ using (C3). Employing the renormalized equations of motion we arrive again at (C6), but where all the transport coefficients are now primed. We expect entropy to still be conserved upon integrating out the fast modes via DRG, which means that Eq. (C6) should vanish identically even for the primed values of the transport coefficients. This immediately severely constrains the parameters, forbidding graphical corrections to the nonlinear terms.

As an example, we can look at the term $\partial_\parallel |\vec{v}_\perp|^2$ in (C6). The associated coefficient is

$$\begin{aligned} 2f'_3 - v_0 \rho_0 - 2\mu_0 w'_3 &= 2f_3 + 2\delta f_3 - v_0 \rho_0 - 2\mu_0 w_3 \\ &\quad - 2\mu_0 \delta w_3 = 2\delta f_3 - 2\mu_0 \delta w_3, \end{aligned} \quad (C9)$$

where in the last step we used the fact that the bare theory (without the prime) identically solves term by term Eq. (C6). Hence, the requirement that entropy remains conserved along the RG flow implies that this term vanishes identically, namely, $\delta f_3 - \mu_0 \delta w_3 = 0$. Because the chemical potential μ_0 is a model-dependent function which never appears in our dynamical Eqs. (C4), the only possibility is that the two graphical corrections δf_3 and δw_3 vanish separately at all orders in perturbation theory.

The same logic applies to all the terms in (C6) up to order three in fluctuations, suggesting that all the linear and nonlinear terms with only one derivative never receive graphical renormalization. Consequently, the scalings in (15) are exact at all orders, which is the conclusion of our argument. Notice

⁴We list here only the nonlinear transport coefficients; however, in principle, also v_0 , v_2 , κ , and n_0 in (C4) could receive corrections. Our argument implies that these terms too do not receive perturbative corrections, as can be argued also using diagrams [3].

that higher-order dissipative corrections [the ξ and D terms in (13a) and (13b)] are not constrained by this argument, as it should be, since for them the second law of thermo-

dynamics does not impose any equality. Indeed, it is well known that viscous terms do receive perturbative corrections under DRG [3,34].

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