Università degli Studi di Genova

SCUOLA DI SCIENZE MATEMATICHE, FISICHE E NATURALI
Corso di Laurea Magistrale in Fisica

Tesi di Laurea Magistrale

Anomalous hydrodynamic description of thermoelectric transport in a Weyl semimetal

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Anno Accademico 2019-2020

Abstract

Hydrodynamics is an effective many-body field theory at finite temperature that can be used to describe the long scale (both in space and time) macroscopic dynamics of a system, close to thermodynamic equilibrium. The hydrodynamic description becomes relevant when the system is strongly coupled: in this regime the quasiparticles that govern the dynamics are strongly interacting and short lived, so it is not possible to study these phases in terms of the quasiparticles-like degrees of freedom. Hydrodynamics, on the other hand, does not rely on the existence of quasiparticles and it focuses on the quantities that are (almost)-conserved on long scale, such as energy, momentum and charge.

Hydrodynamics has been recently applied to the study of strongly coupled systems in condensed matter physics, such has high-temperature superconductors and strange metal phases. In this work we are going to use the theory of hydrodynamics to study the thermoelectric properties of a Weyl semimetal when also a small magnetic field is applied.

Contents

Chapter 1

Introduction

From a modern perspective hydrodynamics is defined as a many-body effective field theory at finite temperature that describes the long range dynamics of systems close to thermodynamic equilibrium; it has been recently applied with good success to the study of the transport properties of strongly correlated systems. In this Master Thesis we are going to apply the theory of relativistic hydrodynamics to study the thermoelectric transport properties of a Weyl semimetal.

The microscopic dynamics of many physical systems can be described in terms of quasiparticles, this often happens when the quasiparticle-like degrees of freedom are weakly interacting. When the quasiparticles are weakly interacting the scattering cross section is small and thus the lifetime of the quasiparticles is very long (the mean free path l_{mfp} depends on the density and the cross section $l_{\text{mfp}} \sim (\rho\sigma)^{-1}$). In this situation it is possible to describe the system using methods that are based on the existence of long-lived quasiparticles, such as the Fermi liquid, kinetic theory or quantum field theory perturbative approaches; a prime example of these systems is the metal phase in condensed matter physics, whose description is based on the Fermi liquid. In a metal, for example, the electrons are about 2 Å apart, but they are so weakly interacting that the mean free path is longer than 10^4 Å at room temperature [1]. Furthermore, in the case of the electron fluid in condensed matter, energy and momentum are usually not conserved: electron-impurity and Umklapp scattering violate momentum conservation and electron-phonon scattering degrade both momentum and energy [2].

There are however certain phases in which the quasiparticles are strongly coupled. When this happens, the lifetime of the quasiparticles becomes

the shortest time scale of the system and the quasiparticles description cannot be applied. In this regime the only quantities that are (almost)-conserved on long time scale are energy, momentum and charge. In this regime the quasiparticles interact so frequently that the system reaches a local thermodynamic equilibrium condition very quickly and an hydrodynamic description is possible.

Hydrodynamics thus studies only the long-range (long wavelength), long-time (small frequency) scale evolution of a system near local thermodynamic equilibrium, it relies only on the symmetries of the system and does not depend on the specific details of the microscopic dynamics.

Hydrodynamics has been used successfully to study the transport properties of many strongly interacting systems in condensed matter, such as Weyl semimetals [3, 4], graphene [5] but also the strange metal phases of High-Temperature Superconductors [6, 7]. The physics of a hydrodynamic electron fluid is different from the kinetic regime and there are distinct signature to look for in experiments; recently some experiments managed to test some of these characteristic features [8–11].

Hydrodynamics is also used to study other strongly coupled systems such as the quark-gluon-plasma phase of the QCD, but also in astrophysics and cosmology [12, 13].

There are also theoretical interests in hydrodynamics, approaching hydrodynamics as an effective field theory near thermodynamic equilibrium, furthermore the fact that the hydrodynamic regime partially overlaps with the holographic regime provides a useful tool to study the holographic correspondence.

In this work, as we mentioned, we are going to apply the theory of relativistic hydrodynamics to model the macroscopic dynamics of a Weyl semimetals. Weyl semimetals are a class of materials first predicted to exist in the '30, but only very recently found in experiments [14]. The semimetals are characterized by the fact that the conduction and the valence bands become degenerate in specific points, called Weyl nodes. Close to these points, where also the Fermi energy lies, the band structure is such that quasiparticles that govern the dynamics of the semimetal are described by the Weyl equation. This indicates that the quasiparticles are massless fermions with distinct chirality that also carry electric charge. The Weyl nodes also inherit the distinct chirality property, so that there is always an even number of Weyl nodes, half are left-handed and half right-handed.

According to classical field theory, a theory of massless fermions should

present a conserved axial current (that, together with the conserved total current, implies the conservation of the chiral currents), however this is not true at the quantum level anymore. In quantum field theory the chiral symmetry is not an exact symmetry and it leads to a so called quantum anomaly; it appears as a correction to the conservation equation for the axial current as an extra term that depends on the gauge fields.

From a practical point of view, what the anomaly tells us it that in classical field theory the number of left and right-handed particles is conserved, but at the quantum level, when an electric and magnetic field parallel to each other are applied to the system, there is an uneven production of left and right-handed particles, such that the total current is conserved, but the axial current is not.

The anomaly also has significant implications on the phenomenological macroscopic transport properties of chiral fluids, such as the chiral magnetic effect and the chiral vortical effect. Because of the axial anomaly the Weyl semimetals also have a so called negative magnetoresistance, this means that when an electric and magnetic field parallel to each other are applied to the semimetal, the longitudinal DC electrical conductivity grows with the magnetic field [15].

Weyl semimetals are usually characterized in term of the microscopic quasiparticle dynamics, so that the hydrodynamic description is not applicable. In certain regimes, however, when the chemical potential is close to the Weyl nodes, the interaction between the quasiparticles can become very strong; in this situation hydrodynamics can be used to study the transport properties of the Weyl semimetal.

In the model we considered the Weyl semimetal as described by a vector current (that is exactly conserved) and an axial current (that is not conserved because of the chiral anomaly), we also applied to the system a constant magnetic field B. Working in linear response theory, we found the equilibrium solution to the hydrodynamic equations and from there we slightly perturbed the system with a small electric field and temperature gradient. We also added small dissipation terms in the equations of linearized hydrodynamics, these terms are needed to have a finite DC longitudinal conductivity (in our model we are explicitly breaking the spacetime translations and the chiral symmetry with a small parameter). Following the procedure by Martin and Kadanoff [16] we manage to compute the full thermoelectric matrix for the system. The results we obtained for the transverse coefficients are in agreement with the expressions found in literature, but we generalize the

result for a system described by two currents. The longitudinal anomalous conductivities however (particularly those induced by a temperature gradient), still show some problematic features in our model. In particular the DC conductivities are not all bounded and, furthermore, they are not Onsager reciprocal, so we need to impose external constraints on the thermodynamics parameters.

In chapter 2 we briefly explain the main features of the Weyl semimetals and we explain the role of the chiral anomaly and the negative magnetoresistance, in chapter 3 we review the classical theory of relativistic hydrodynamics and the linear response theory, in chapter 4 we modify the equations of hydrodynamics to account for the chiral anomaly and we perform the linear response theory computation to obtain the full thermoelectric magneto-transport matrix, in chapter 5 we discuss the results obtained and we give an outlook on possible future studies on the the chiral fluids.

Chapter 2

Weyl semimetals

Contents

2.1 Introduction

In 1929 Weyl came up with a simplified version of the famous Dirac equation, that is used to describe the wave function of massive fermions in a relativistic setting. Weyl equation, on the other hand, describes massless fermions with distinct chirality. The Standard Model of particle physics does not contain any massless fermion and as far as we know there are no fundamental particles of this kind, however the Weyl fermions can appear as quasiparticles (i.e. low-energy excitations) in condensed matter systems. Weyl semimetals are thus a prime example to study such particles in the real world, but they also present peculiar features that exists only in the condensed matter context.

In condensed matter physics usually the energy scales are much smaller then the electron mass, so a relativistic description seems to be unnecessary. However, the propagation of even slow electrons through the periodic potential

of a crystal leads to a dressing of the electronic state, so that in certain circumstances the low-energy behaviour of the dynamics can be described with Weyl and Dirac equation. A famous example in this regard is the graphene, where a linear dispersion relation is captured by the massless two-dimensional Dirac equation.

In 1937 Herring investigated the conditions needed to have degenerate electronic band structure and from there he found that there could be twofold degeneracies in three-dimensional materials that could be described by the Weyl equation. The contact points are called Weyl nodes (or sometimes Weyl points) and the linear band structure around the nodes identifies a Weyl valley (or Weyl cone). Weyl semimetals are a class of materials in which the chemical potential is near the Weyl nodes, so that the low-energy excitations behave like massless fermions. Many physical properties of the Weyl fermions continue to hold in this non-relativistic condensed matter framework, such as the chiral anomaly.

Weyl semimetals also have many topological properties, but we will not go in the details here. For completeness, however, we will indicate the two most important features. The first one is the presence of a Fermi arc: the surface states do not form a closed Fermi surface, like in the bulk, instead they form an arc that is bounded by the projections to the surface of a pair of Weyl nodes (Figure 2.1). The second important topological feature is the fact that Weyl nodes always come in pairs, half that contain right-handed particles, half with left-handed particles. Naively we can justify this last claim by observing that otherwise the chiral anomaly would imply the non conservation of the total electric charge (we will explain this better in the following).

2.2 Weyl equation

In this chapter we will be working with metric $\eta_{\mu\nu} = \text{diag}(+1, -1, \dots, -1)$, natural units $\hbar = c = 1$ and d spatial dimensions (unless otherwise stated). Greek indices run from 0 to 3, while Latin indices from 1 to 3.

The famous Dirac equation is usually written in the form

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

where the γ^μ are $d+1$ matrices that satisfy the anticommutation relations $\{\gamma^\mu, \gamma^\nu\} = 0$ for $\mu \neq \nu$ and $(\gamma^0)^2 = -(\gamma^i)^2 = \mathbb{1}$. ψ is a $d+1$-dimensional spinor.

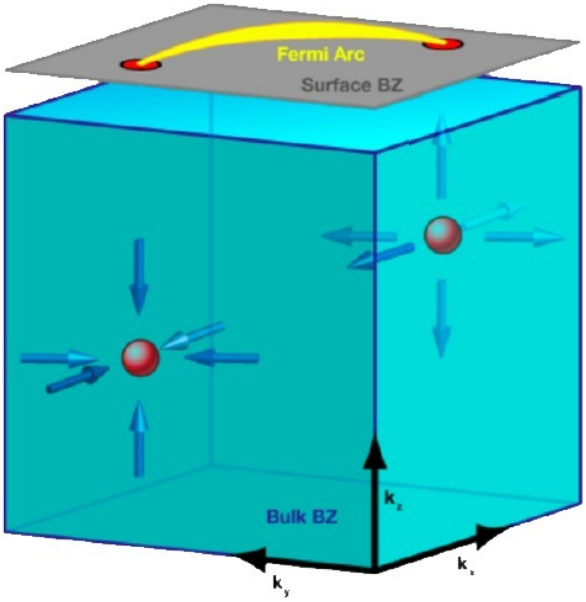

Figure 2.1: Image from [14]. A schematic figure of a bulk Brillouin zone with a Fermi arc on the surface that connect the projection of the two nodes.

Weyl noticed that this equation can be further simplified in an odd number of spatial dimensions. In $d = 3$ one possible representation of the algebra is the so called Weyl or chiral representation, in this case the γ^μ are 4×4 block matrices of the form

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

with σ^i the 2×2 Pauli matrices σ_x, σ_y and σ_z. Because the Lorentz group is reducible, we can decompose the 4-dimensional spinor ψ in 2-dimensional blocks. Writing $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$ we define ψ_L and ψ_R the left-handed and right-handed Weyl two components spinor.[1] With this definition the Dirac equation can be written as

$$(i\gamma^\mu \partial_\mu - m)\psi = \begin{pmatrix} -m & i(\partial_t + \boldsymbol{\sigma} \cdot \boldsymbol{\nabla}) \\ i(\partial_t - \boldsymbol{\sigma} \cdot \boldsymbol{\nabla}) & -m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0$$

If the mass term is zero, then we can decompose the system in two separate

[1] Some texts prefer to use the notation ψ_+ and ψ_- for the left and right-handed particles respectively.

equations for left and right-handed spinors [17]

$$i(\partial_t - \boldsymbol{\sigma} \cdot \boldsymbol{\nabla})\psi_L = 0 \tag{2.1a}$$

$$i(\partial_t + \boldsymbol{\sigma} \cdot \boldsymbol{\nabla})\psi_R = 0 \tag{2.1b}$$

these are the so called Weyl equations.

In any odd spatial dimension space $d = 2k + 1$ we can also define the Hermitian matrix $\gamma^5 = i^k \gamma^0 \gamma^1 \dots \gamma^d$, which satisfies $\{\gamma^5, \gamma^\mu\} = 0$ and $(\gamma^5)^2 = \mathbb{1}$. With this matrix we can create a projection operator $P_{L/R} = \frac{1}{2}(\mathbb{1} \pm \gamma^5)$ (a projector satisfies $P_{L/R}^2 = P_{L/R}$ and $P_L P_R = 0$) such that

$$\psi_{L/R} = P_{L/R}\psi$$

ψ_L and ψ_R spinors are the chiral components of the ψ Dirac spinor. Chirality is a mathematical property of the particle fields and it determines whether the field transforms in a left or right-handed representation of the Poincaré group. In the case of massive particles, chirality is a disconnected definition from helicity (the projection of the spin in the direction of the momentum, in operators $\hat{h} = \frac{1}{2}\hat{\mathbf{p}} \cdot \boldsymbol{\sigma}$): helicity is not Lorentz invariant (with a boost the momentum can change sign, while the spin remains constant) but is conserved in a given reference system.

For massless particles, however, chirality and helicity identify the same property: right-handed particles are particles that have the spin parallel to the momentum, while left-handed particles have the spin anti-parallel to the momentum (massless particles travel at the speed of light and it is not possible to perform a boost that changes the sign of the momentum). This can be seen noting that with the momentum operator $\mathbf{p} = -i\boldsymbol{\nabla}$ the Weyl equations become

$$i\partial_t \psi_L = -\mathbf{p} \cdot \boldsymbol{\sigma} \psi_L$$

$$i\partial_t \psi_R = \mathbf{p} \cdot \boldsymbol{\sigma} \psi_R$$

so that it is now explicit that left-handed particles have $-1/2$ helicity and right-handed particles $+1/2$.

2.3 Band structure

In a Weyl semimetal the conduction and valence bands become degenerate (i.e. the two bands have the same energy) in specific points of the Brilluoin zone and this band structure is stable to small variation of the parameters.

If spin rotation symmetry is assumed (e.g. ignoring the spin-orbit couplings), then the various bands are doubly degenerate. To break this degeneracy the usual way is to have either time-reversal or inversion symmetry ($\mathbf{r} \to -\mathbf{r}$) broken in the system (both these symmetries reverse the momentum). This ensures that only the points for which $\mathbf{k} = -\mathbf{k}$ are still degenerate. The time-reversal symmetry can be broken by a magnetic field, the persistence of the inversion symmetry ensures that the two Weyl nodes are at the same energy [14].

After we break one of the two symmetries we are left with a two levels non-degenerate system. We can start by studying the conditions to have accidental degeneracies in a simple two level structure. The most general 2×2 Hamiltonian we can make is given by a linear combination of the Pauli matrices σ_i and the identity

$$H(\mathbf{k}) = f_0(\mathbf{k})\mathbb{1} + f_1(\mathbf{k})\sigma_x + f_2(\mathbf{k})\sigma_y + f_3(\mathbf{k})\sigma_z$$

The energy level splitting is given by $\Delta E = 2\sqrt{f_1^2 + f_2^2 + f_3^2}$ (f_0 does not appear, since it produces an overall constant for the two levels), so to have degeneracy $\Delta E = 0$ we need to tune all three parameters to zero simultaneously $f_1 = f_2 = f_3 = 0$ and to do so we need, in general, three independent variables, i.e. we are in three spatial dimensions.

A simple argument can be used here to give intuition about why a system with Weyl nodes is stable to small perturbations of Hamiltonian parameters. We can consider a real function $f_1(\mathbf{k})$: the equation $f_1(\mathbf{k}) = 0$ identifies a closed surface that separate positive and negative value of f_1. By asking that also $f_2(\mathbf{k})$ and $f_3(\mathbf{k})$ vanish simultaneously, the intersection of three surfaces will in general occur at isolated points, the Weyl nodes. We now consider a small perturbation of one of the functions f_i, this will in general change the position of the intersection point, but the three surfaces will still have one contact point. The only way to remove the Weyl nodes is by annihilation with another Weyl nodes. Intuitively, if the $f_i = 0$ surfaces are (almost) spherical, then they come in touch in 2 points; by moving one of the surface the two points come closer and closer until they reduce to a single point (a Dirac node) and after that there is no more touching between the three surfaces.

The above discussion is quite general and the tuning of the three functions is needed to have the degeneracies that produce the Weyl nodes. In a Weyl semimetal, however, we also require that the the nodes have to be close to the Fermi energy. Following the formalism above, this means that $f_0(\mathbf{k})$ also has to be nearly zero.

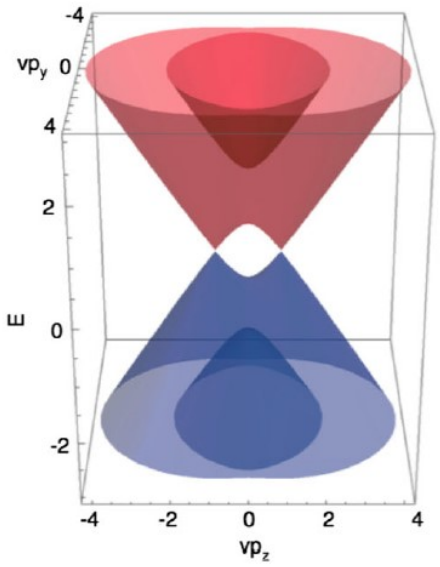

Figure 2.2: Image from [14]. A representation of two Weyl nodes with their respective cones with linear dispersion relation.

We now focus on the structure close to the touching point. If $\mathbf{k}_0$ is the position of the point, then we can expand the Hamiltonian above about $\mathbf{k} = \delta\mathbf{k} + \mathbf{k}_0$. At first order we find

$$H(\mathbf{k}) \sim f_0(\mathbf{k}_0)\mathbb{1} + \mathbf{v}_0 \cdot \delta\mathbf{k}\mathbb{1} + \sum_{a=x,y,z} \mathbf{v}_a \cdot \delta\mathbf{k}\sigma^a$$

where we defined $\mathbf{v}_\mu = \boldsymbol{\nabla}_\mathbf{k} f_\mu(\mathbf{k})|_{\mathbf{k}=\mathbf{k}_0}$ that are the effective velocities of the quasiparticles close to the node and are, in general, different. If we reduce to the special limit where $\mathbf{v}_0 = 0$ and there is isotropy $\mathbf{v}_a = v_F\hat{\mathbf{a}}$ (with $a = x, y, z$) then Schrodinger equation reduce to Weyl equation (2.1). The chirality of the point is determined by $C = \text{sign}(\mathbf{v}_x \cdot \mathbf{v}_y \times \mathbf{v}_z)$ and v_F (the Fermi velocity) is the effective velocity that takes the role of c in the relativistic theory of massless fermions.

2.4 Chiral anomaly

Anomalies, heuristically speaking, arise whenever a symmetry of a classical theory is not a symmetry of the quantum theory: because of the radiative corrections the currents that would otherwise be conserved gain an extra term that breaks the conservation law.

Consider a classical QED theory with just one fermion family, described by the Lagrangian density

$$\mathcal{L} = \bar{\psi}(i\not{D} - m_0)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

where $D_\mu = \partial_\mu - ieA_\mu$ is the usual QED covariant derivative (e is the coupling constant) and m_0 is the bare mass of the fermion. This theory is invariant under a $U(1)$ global symmetry, changing the phase of the fermion field

$$\psi \to e^{-i\alpha}\psi \qquad \text{and} \qquad \bar{\psi} \to e^{i\alpha}\bar{\psi}$$

with α some constant (it is actually invariant under the gauge symmetry, with $\alpha = \alpha(x)$, that is a much stronger constraint). This symmetry implies by Noether's theorem that there is a conserved vector current

$$J^\mu = \bar{\psi}\gamma^\mu\psi \qquad \Longrightarrow \qquad \partial_\mu J^\mu = 0$$

We could also do a chiral transformation on the fields, in which left and right-handed fermions receive unequal phases (this is because $\{\gamma^\mu, \gamma^5\} = 0$)

$$\psi \to e^{-i\alpha\gamma^5}\psi \qquad \text{and} \qquad \bar{\psi} \to e^{-i\alpha\gamma^5}\bar{\psi}$$

This transformation in general is not a symmetry of the theory, unless the fermions are massless. The axial current obeys

$$J_5^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi \qquad \Longrightarrow \qquad \partial_\mu J_5^\mu = 2im_0\bar{\psi}\gamma^5\psi$$

The above expressions are correct in classical field theories: if the fermions are massless, both the vector current J^μ and the axial current J_5^μ are exactly conserved. These two symmetries together imply the separate conservation of left and right-handed fermion numbers and a similar result is obtained in QCD.

However at the quantum level, in gauge theories, the conservation of the axial vector current is incompatible with gauge invariance (in an even-dimensional spacetime, so $1+1$ or $3+1$); the radiative corrections imply that the axial current is not conserved and it gains an extra term that contains the gauge field. This is the so called axial ABJ anomaly (or chiral anomaly, since an anomaly on J_5^μ implies an anomaly on $J_{R/L}^\mu$ or sometimes even triangle anomaly, from the form of the Feynman diagram that is used to study the anomaly) after Adler-Bell-Jackiw who first studied this effect in 1969. Another way to better explain this concept is that the chiral symmetry

is a symmetry of the (classical) action (for massless QED), but not of the effective renormalized action. The equation that describes the divergence of the axial current with the anomaly becomes [17]

$$\partial_\mu J_5^\mu = \frac{e^2}{16\pi^2}\epsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu} \tag{2.2}$$

Usually this result is proved from the explicit calculation in perturbation theory, by showing that the divergence of the axial current has a nonzero matrix element to create two photons at the one loop level, or by analyzing the conservation law from the functional integral of the fermion field. The two-dimensional case is, instead, given by the expressions

$$\partial_\mu J_5^\mu = \frac{e}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu}$$

where it is important to note that, even if $\epsilon^{\mu\nu}$ appears, in $1+1$ dimension the expression $\epsilon^{\mu\nu}F_{\mu\nu}$ is related to the electric field, not the magnetic field.

This result also has a global interpretation that is probably easier to understand. Integrating over the spacetime volume the divergence of the axial current gives, in absence of chiral anomalies (in two dimensions)

$$\int \mathrm{d}^2x\, \partial_\mu J_5^\mu = N_R - N_L = 0$$

With N_R and N_L the right and left-handed chirality particle numbers. If we add the anomaly this difference is not conserved anymore and it does depend on the gauge field $F^{\mu\nu}$: what happens is that because of the gauge $U(1)$ field the production rate of right and left-handed particles is not equal and more right (or left)-handed particles are created, creating an imbalance in the axial current [15].

In a Weyl semimetal with a pair of Weyl nodes of opposite chirality, the number of electrons in the vicinity of each node is modified in the presence of electric $\mathbf{E}$ and magnetic field $\mathbf{B}$ via (in three spatial dimensions)

$$\frac{\mathrm{d}\rho_{R/L}}{\mathrm{d}t} = \pm\frac{e^2}{h^2}\mathbf{E}\cdot\mathbf{B} \tag{2.3}$$

where the specific value of the constant depends on the dimensionality of the system. This is another representation of the chiral anomaly and it implies that, even with spatially uniform fields (that can be oriented in an arbitrary direction relative to the separation of the Weyl nodes), the density of electrons in each individual node is not conserved.

This immediately shows why, as we mentioned earlier, a system with a set of uneven Weyl node of left and right-handed chirality is problematic, because

it leads to non conservation of the electric charge $\rho = \rho_R + \rho_L$. If instead the net chirality is zero (half left-handed and half right-handed nodes), then there is no net total charge produced and the nodes act as sources and sinks of electrons.

The formal derivation of the chiral anomaly is rather difficult, but we can give here a more intuitive explanation for this result. We start from the $1+1$-dimensional Weyl equation and then we show how to generalize to the $3+1$ case. We will follow the explanations from [18] and [15].

In $1+1$ dimension there are only two Clifford algebra matrices and a possible representation is given by the Pauli matrices.

$$\{\gamma^\mu, \gamma^\nu\} = \eta^{\mu\nu} \qquad \Longrightarrow \qquad \gamma^0 = \sigma^1 \quad \text{and} \quad \gamma^1 = i\sigma^2$$

The massless Dirac spinor is then a two-component object whose free action is

$$S = \int \mathrm{d}^2x \; i\bar{\psi}\gamma^\mu \partial_\mu \psi$$

In contrast to the $3+1$ dimensions case, these particles have no internal spin, because there is no spatial rotation group in $1+1$ dimensions. We can write the action as

$$S = \int \mathrm{d}^2x \; i\psi^\dagger \gamma^0(\gamma^0\partial_t + \gamma^1\partial_1)\psi = \int \mathrm{d}^2x \; \psi^\dagger(\partial_t - \gamma^5\partial_x)\psi$$

where we defined

$$\gamma^5 = -\gamma^0\gamma^1 = -i\sigma^1\sigma^2 = \sigma^3$$

We can again use the γ^5 matrix to decompose the massless Dirac spinor in its chiral constituents as we showed for the $3+1$ case.

$$\psi_{L/R} = \frac{1}{2}(\mathbb{1} \pm \gamma^5)\psi \qquad \Longrightarrow \psi_L = \begin{pmatrix} \chi_L \\ 0 \end{pmatrix} \quad \text{and} \quad \psi_R = \begin{pmatrix} 0 \\ \chi_R \end{pmatrix}$$

With this decomposition the action becomes

$$S = \int \mathrm{d}^2x \; i\chi_L^\dagger(\partial_t - \partial_x)\chi_L + i\chi_R^\dagger(\partial_t + \partial_x)\chi_R$$

and from here we easily understand that the χ_L has equation of motion $(\partial_t - \partial_x)\chi_L = 0$ with solution $\chi_L = \chi_L(t+x)$, so it is a left-moving fermion; in contrast, $\chi_R = \chi_R(t-x)$ is a right-moving fermion.

After the quantization both particles and anti-particles appear, right-movers have obviously momentum $p > 0$, while left-movers have $p < 0$, but the dispersion relation is $E = |p|$ for all the excitations.

The action above has two global symmetries which rotate the individual phases of χ_L and χ_R. From Noether's theorem this means that the number N_L of left-moving fermions and N_R of right-moving ones is separately conserved, this is referred to as the chiral symmetry. If we deform the theory, provided that both symmetries are preserved, we would expect to not be able to change a left-moving fermion in a right-moving one.

We now couple this $1+1$ massless fermions theory to an external non-dynamical gauge field A_μ by means of minimal coupling. The action now becomes

$$S = \int \mathrm{d}^2x \; i\bar{\psi}\gamma^\mu D_\mu \psi \tag{2.4}$$

with $D_\mu = \partial_\mu - ieA_\mu$ the covariant derivative. This new action is still invariant under the two global symmetries that rotate the phases and naively we would think that N_L and N_R are still separately conserved. We set ourselves in the vacuum state and we consider two $1+1$-dimensional Weyl nodes of opposite chirality. All the negative energy states are filled and we turn on an electric field $\mathcal{E} > 0$ for a time t (the field points towards the right side). Because the particles are charged, the electric field increases the momentum p (and so the energy E) of all the particles of the filled bands by

$$\Delta k = e\mathcal{E}t$$

Both left and right-movers get shifted by the same amount (notice that a hole in the filled band behaves as a particle of opposite momentum) producing the Fermi surface in Figure 2.3. The particles of each chirality are not conserved: we started from a vacuum state, but now we have right-moving particles and left-moving antiparticles.

Denote by ρ_R the density of right-moving fermions and by ρ_L the density of left-moving ones. The shift in momentum becomes a shift in charge density

$$\rho_R = \frac{e\mathcal{E}t}{2\pi} \qquad \text{and} \qquad \rho_L = -\frac{e\mathcal{E}t}{2\pi}$$

where the factor $1/2\pi$ comes from the density of states. The total charge (counting, as usual, the number of particles minus anti-particles) is conserved $\dot{\rho} = 0$ with $\rho = \rho_R + \rho_L$, but the axial charge, i.e. the difference between fermion numbers, is not conserved

$$\frac{\mathrm{d}\rho_A}{\mathrm{d}t} = \frac{e\mathcal{E}}{\pi} \qquad \text{where} \quad \rho_A = \rho_R - \rho_L \tag{2.5}$$

This is the so called axial anomaly in $1+1$ dimension.

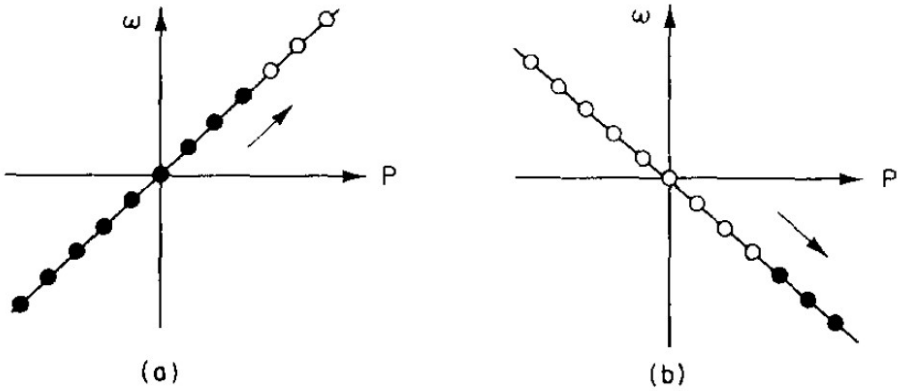

Figure 2.3: Image from [15]. Two Weyl nodes of opposite chirality. The vacuum state is when the bands are totally filled up to energy $\omega = 0$. An electric field produces a shift in the momentum of the fermions: the (a) right-handed fermions will move to the right because of the constant electric field (with positive electric charge). The (b) left-handed fermions with negative energy also move to the right, to gain momentum, but this produces left-handed antiparticle excitations.

We can now explain the anomaly in the $3 + 1$ case. Intuitively what happens is that because of the magnetic field the energy levels of the system become the Landau levels. The lowest Landau level is chiral and, when an electric field is applied to the system, the same effect we just showed for the $1 + 1$ case happens.

We can start considering a single Weyl node of negative chirality, minimally coupled to a magnetic field $\mathbf{B}$. In this case, with $A_0 = 0$ (the electric field is still off), the Weyl equation becomes

$$i\partial_t \psi_L = i\sigma^i D_i \psi_L$$

where $D_i = \partial_i - ieA_i$ is again the covariant derivative. We can write the Hamiltonian for this Weyl equation as

$$H = i\sigma^i D_i = -(\mathbf{p} - e\mathbf{A}) \cdot \boldsymbol{\sigma}$$

Squaring the Hamiltonian, and using the fact that $\sigma^i \sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k$, we find

$$H^2 = (\mathbf{p} - e\mathbf{A})^2 - 2e\mathbf{B} \cdot \mathbf{S}$$

where $\mathbf{S} = \frac{1}{2}\boldsymbol{\sigma}$ is the spin operator (even if for massless particles it is better to refer to this as the helicity). The first term is the usual Hamiltonian

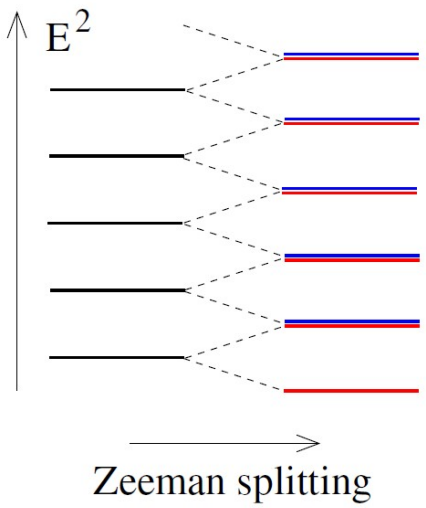

Figure 2.4: Image from [18]. The energy levels of the system (for a fixed p_z) where we can see the degeneracy created by the coupling of the spin with the magnetic field.

for a non-relativistic particle in a magnetic field, while the second term is responsible for the Zeeman splitting between spin states.

We now pick the magnetic field to lie along the z direction $\mathbf{B} = (0, 0, B)$ and choose to work in the Landau gauge, so we have $\mathbf{A} = (0, Bx, 0)$. The above equation becomes

$$H^2 = p_x^2 + (p_y - eBx)^2 + p_z^2 - 2eBS_z$$

The quantization in the xy plane leads to the usual Landau levels. They have a large degeneracy that depends on the area of the plane, the number of state is given by $eBA/2\pi$; this is because (in this gauge) the Landau levels do not depends on the quantum number p_y (the degeneracy is gauge invariant however). The Landau levels also reduce to those of the usual harmonic oscillator. The energy spectrum is then given by

$$E^2 = eB(2n + 1) + p_z^2 - 2eBS_z \qquad \text{with} \quad n = 0, 1, 2, \ldots$$

From this result we can see that the energy levels, for a fixed p_z, are doubly degenerate. Indicating the states with $|n, S_z\rangle$, we see that the state $|n, -1/2\rangle$ is degenerate with the state $|n + 1, +1/2\rangle$ (see Figure 2.4). The only level that is not degenerate is the ground state $|0, +1/2\rangle$ with energy $E = 0$ (this level has half the states of the other levels), called Landau zero mode.

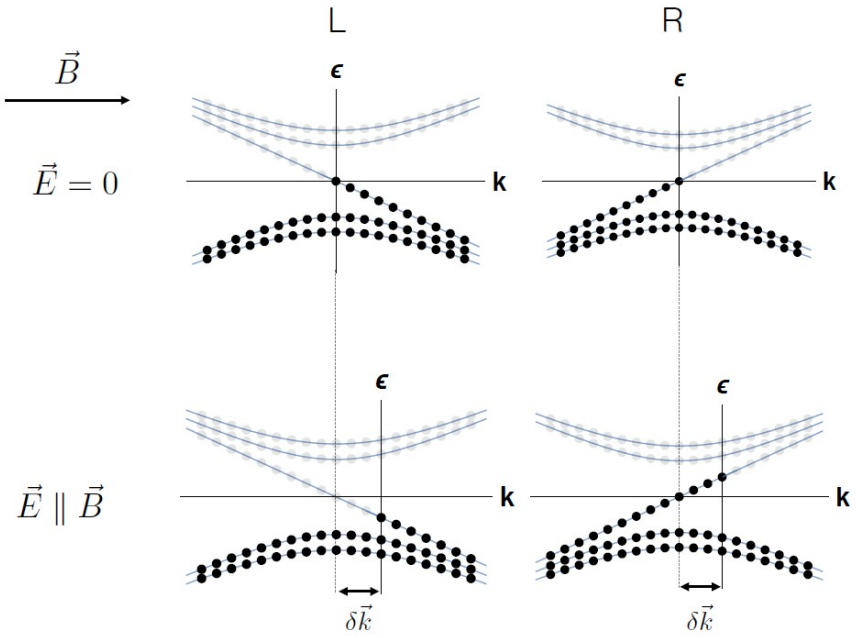

Figure 2.5: Image from [19]. On top, the effect of the magnetic field that creates the two chiral zero modes in the nodes of opposite chirality. On the bottom, the shift in momentum that happens because of the electric field.

If the magnetic field is strong enough we can focus just on this lowest Landau level at $n = 0$. This level is filled with particles with spin $+\frac{1}{2}$, so they have the form

$$\psi_L(x, y; z, t) = \begin{pmatrix} \chi_L(x, y; z, t) \\ 0 \end{pmatrix}$$

where the notation helps us to remember that the xy dependence is fixed by the Weyl equation and the Landau modes, while the zt dependence is still not fixed. By plugging this ansatz back in the action we find

$$S = A \int \mathrm{d}z\, \mathrm{d}t\; i\bar{\chi}_L(\partial_t - \partial_z)\chi_L \tag{2.6}$$

and we see that the zero modes that come from χ_L are left-movers in the z direction (with spin up antiparallel to the direction of motion, so with negative chirality as we claimed). A is a constant that includes the energy content of the xy part of the wave function.

Applying the same procedure for the ψ_R chiral fermions in a right-handed Weyl node produce again Landau zero modes, but this time the effective

action that describes the motion on the zt plane has a flipped sign and so the particles are right-movers.

We can now reintroduce in the effective action (2.6) the background gauge fields

$$S = A \int \mathrm{d}t\, \mathrm{d}z\; i\chi_L^\dagger D_- \chi_L + i\chi_R^\dagger D_+ \chi_R$$

where we defined $D_\pm = D_t \pm D_z$ and D_μ is again the covariant derivative that couples these chiral zero modes to the two gauge fields A_0 and A_z.

We recognize that this effective action has the same form of (2.4), that describes two-dimensional massless fermions coupled to an electromagnetic field and that, as we already showed, does not have a conservation law associated with the chiral symmetry. We can thus use the result (2.5) for the $1+1$-dimensional chiral anomaly to also describe the effect of the anomaly in four dimensions, all we need to do is to take care of the lowest Landau level degeneracy. The lowest Landau level, that is the one we are interested in, has a density of state per area given by $eB/2\pi$ and each of these state contributes to the anomaly. In $3+1$ dimensions, the axial charge changes if we turn on both a magnetic field B and an electric field $\mathcal{E}$ lying in the same direction (the electric field has to act on the reduced $2D$ action, so it points towards the z direction, like B does). This produces a total axial density rate of

$$\frac{\mathrm{d}\rho_A}{\mathrm{d}t} = \frac{eB}{2\pi}\frac{e\mathcal{E}}{\pi} = \frac{e^2}{2\pi^2}\mathbf{E}\cdot\mathbf{B} \tag{2.7}$$

that is the same result we claimed for the Quantum Field Theory equation and agrees with (2.3) (once we revert to SI units).

Very briefly, this process can be described by saying that the magnetic field forces all the fermions to align their spins towards the z axis (basically forcing the system to have only one relevant dimension), while the electric field produces a momentum imbalance between particles of different chirality.

In a purely quantum field theoretical approach the production of fermions of opposite chirality is, usually, described in term of the Dirac sea. The same discussion we just showed for a single Weyl node with a filled valence band applies exactly the same, thinking the Dirac sea with all negative energy states completely filled. In the condensed matter context, however, we have to explain where these newly created fermions come from.

There are two main sources that inject and destroy fermions in the Weyl semimetals: the first one is normal inter-valley scattering, so that a scattering event between fermions that come from Weyl nodes of opposite chirality can

produce a density relaxation

$$\frac{\mathrm{d}\rho_A}{\mathrm{d}t} = \frac{e^2}{2\pi}\mathbf{E}\cdot\mathbf{B} - \frac{\rho_A}{\tau_c}$$

where τ_c is a time constant that disperses axial charge and that depends on the specific of the scattering mechanism. The other source of fermion imbalance is the surface Fermi arc we mentioned earlier in the chapter. An electron that, starting from a Weyl node in the bulk, reaches the surface, can move along the Fermi arc and enter the connected Weyl node of opposite chirality. While this process is very weak in the large volume limit, because of the scaling of the boundary effects with volume, it can be dominant in systems that have weak scattering processes.

2.5 Properties and materials

2.5.1 Negative magnetoresistance

The chiral anomaly discussed above has many effects on physical measurable quantities of a chiral system, such as the chiral magnetic effect (sometimes shortened as CME) and the anomalous Hall conductivity [19]. For a Weyl semimetal, in particular, the chiral anomaly is responsible for an effect called negative magnetoresistance, that is the fact that the longitudinal resistance (i.e. parallel to the magnetic field) has a strong dependence on the magnetic field itself and that, in particular, it decreases as the magnetic field grows (so that the conductivity grows with B). This effect was first predicted in [15] and is a characteristic signature of a Weyl semimetal [14, 20].

Usually the magnetoresistance (MR) is formally defined as a relative variation to the zero-magnetic field value[2], i.e.

$$\mathrm{MR} = \frac{\rho(B) - \rho(0)}{\rho(0)} = \frac{\Delta\rho}{\rho(0)}$$

where $\rho(B)$ is the value of the resistivity with the magnetic field, while $\rho(0)$ is the resistivity without magnetic field applied. In this regard it is now clear why Weyl semimetals are said to show a negative magnetoresistance: since $\rho(B)$ is smaller then $\rho(0)$, from the above definition the magnetoresistance is negative.

The above discussion shows that the axial current is not conserved and that there is a net transport of axial charges in the presence of parallel electric

[2]In certain scenario it is defined with $\rho(B)$ at denominator.

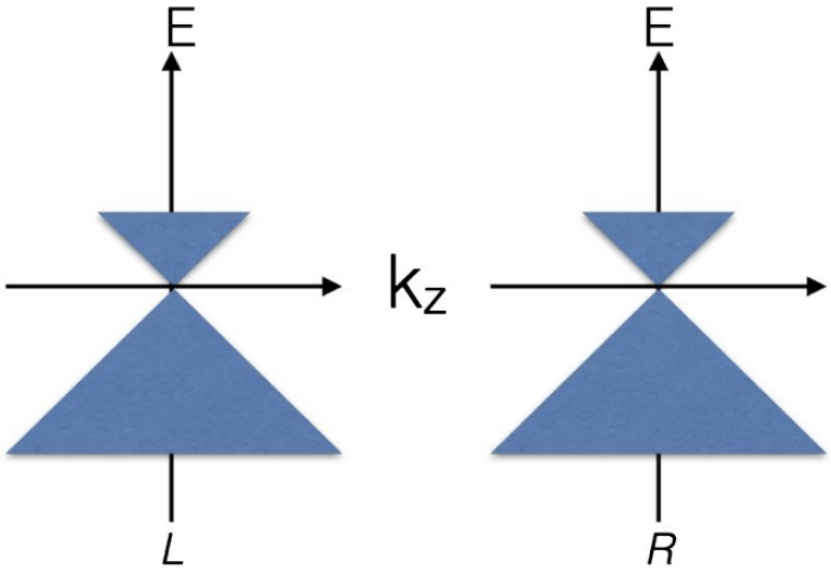

Figure 2.6: Image from [14]. The band filling for a time-reversal symmetry breaking type-I Weyl semimetal. In this case we see that, without external fields, the two chemical potentials for the left and right-handed nodes are at the same level.

and magnetic field. However the axial current is not an observable quantity and, on the other hand, the total electric current is still conserved, even when the anomaly is present, so it is not clear how this could produce measurable effects.

The caveat lies in the fact that in our above (very simplified) derivation we considered the two Weyl nodes to have the same energy: while the result obtained for the axial current is true even in a more general derivation, the electric current also gains an extra term when the chemical potentials of the left and right-handed Weyl nodes are not the same [21, 22]. We can define the axial chemical potential as $\mu_5 = \frac{1}{2}(\mu_R - \mu_L)$ and it is possible to show that, if μ_5 is not zero, the chiral anomaly creates a current [23, 24]

$$\mathbf{J} = c\mu_5\mathbf{B} \tag{2.8}$$

with c a constant that depends on the anomaly. This equation is the basis for the chiral magnetic effect.

Note however that this does not imply the global non-conservation of the electric charge Q, since the resulting total charge flow through a closed

boundary S surrounding the region with $\mu_5 \neq 0$ vanishes

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = -\oiint_S \mathbf{J} \cdot \mathrm{d}\mathbf{S} = 0$$

Usually this result is derived by adding an axion term to the electromagnetic action, i.e. a gauge field that couples to the axial current and to the usual $U(1)$ electromagnetic gauge field, so that the chiral anomaly result (2.2) is obtained by a simple variation of the action.

From this result, we can try to give an intuitive reason from a kinetic approach as to why in a Weyl semimetal the longitudinal electrical conductivity grows with the magnetic field. An effect related to the CME occurs when a density difference of electrons in the two opposite Weyl nodes is created by the chiral anomaly, that can pump charge between nodes in the presence of parallel electric and magnetic fields. This difference in the densities leads to an effective axial chemical potential for the Weyl semimetal that can produce observable consequences of the chiral magnetic effect.

The density difference between the nodes is determined by the balance between the chiral anomaly pumping and the dissipation rate of the inter-valley scattering process (governed by the time constant τ_c).

$$\frac{\mathrm{d}(\rho_R - \rho_L)}{\mathrm{d}t} = \frac{2e^2}{h^2}\mathbf{E} \cdot \mathbf{B} - \frac{\rho_R - \rho_L}{\tau_c}$$

In a steady state these two effects completely cancel out and there is a final state whose densities are proportional to $\mathbf{E} \cdot \mathbf{B}\tau_c$, where we assumed that the intra-valley scattering is much faster then the inter-valley scattering (so a steady state defined by τ_c is a sensible thing to have).

These densities proportional to $\mathbf{E} \cdot \mathbf{B}\tau_c$ are different and also induce an axial chemical potential with the same dependence from the gauge fields, so that the chiral magnetic effect equation (2.8) becomes (focusing on the z direction, in which the magnetic field B lies)

$$J_z \propto (\mathbf{E} \cdot \mathbf{B})\tau_c B$$

From this equation we can already see that the longitudinal DC conductivity, i.e. the proportionality constant between J_z and E_z, depends on the magnetic field squared. Basically this effect derive from two successive uses of the anomaly: first to induce a chemical potential imbalance, then to produce the chiral magnetic effect. The final result for the longitudinal electrical DC conductivity is

$$\sigma(B) = \sigma_0 + \frac{e^2 B^2 \tau_c}{4\pi^4 g(\epsilon_F)} \tag{2.9}$$

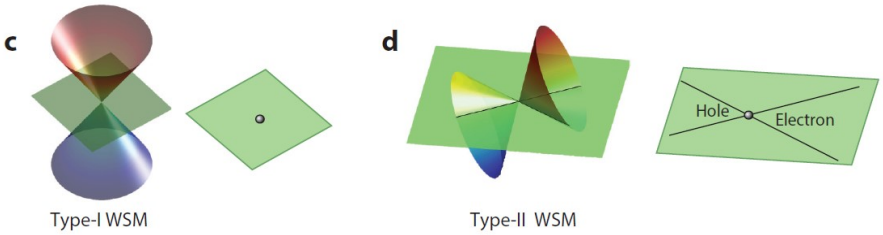

Figure 2.7: Image from [25]. **c**: the typical structure of a type-I WSM; **d**: a type-II WSM. Note that the fermions in type-II WSM manifest Lorentz-violating behaviour [26].

A similar result is obtained for the other thermoelectric transport coefficients: all these coefficients are not independent to each other, instead there are linear relations that link them together (the Ward identities). We can thus expect that all the longitudinal DC thermoelectric conductivities should depend on the square of the applied magnetic field.

A similar result is obtained in hydrodynamics, with the conductivity that grows with the square of the magnetic field (keeping B along the z axis). This quadratic growth is inevitable: the conductivities xx, yy and zz can only depend on B^2 and not on odd powers of B, this is because of the symmetry of the system that requires that $\sigma_{ii}(B) = \sigma_{ii}(-B)$. The same argument requires that the off-diagonal terms xy and yx can only depend on odd powers of B, so that $\sigma_{xy}(B) = \sigma_{yx}(-B)$.

2.5.2 Materials and experimental results

The Weyl semimetals band structure is similar to the one from topological insulators, so the search for such materials often starts by looking for materials with the right crystal structure. As we mentioned, Weyl semimetals need a broken symmetry to destroy the degeneracy of the band structure, so they can be classified in two categories: Weyl semimetals with time-reversal broken symmetry $\mathcal{T}$ (magnetic materials) or with broken inversion symmetry $\mathcal{P}$. They can be further categorized into type-I and type-II WSM: for the type-I WSM the Fermi surface is point-like and it lies in the touching point between the valence and conducion bands, while for type-II WSM the Weyl cones are tilted and so the Fermi surface intersect with the cones (Figure 2.7) [25–27].

It is important to observe that, while the Weyl semimetals are predicted to have a large variety of characteristic effects (anomalous transport, topological properties, peculiar band structure, anomalous Hall effects, ...) it is not

always straightforward to determine whether a material is a Weyl semimetal or not. The negative magnetoresistance is not always conclusive, so better evidences are needed, like the presence of a Fermi arc or a direct check of the band structure.

In 2011 [28] the first materials prediction was in the pyrochlore iridates family $R_2Ir_2O_7$, with R is a rare earth element. These elements are antiferromagnetic, so they break the time-reversal symmetry, however the experimental results are not conclusive about the exact structure of these materials, so it is not totally clear yet if these are actually Weyl semimetals.

A large number of materials that are Weyl semimetals through the inversion symmetry breaking have recently been predicted and discovered, in particular the TaAs (tantalum arsenide) family. TaAs is a material in the type-I class of WSM and in 2015 it was the first known electronic system to be discovered as a WSM [29] (the A-phase of superfluid helium-3 was already known to have Weyl fermions).

Strong evidences for the Weyl semimetals have also been found in other materials of the same class, in particular TaP, NbAs and NbP, that are all type-I inversion symmetry breaking WSMs. For a type-II WSM, theoretically predicted $MoTe_2$ was recently confirmed to be a Weyl semimetal, by direct evidence of the presence of the characteristic Fermi arc.

All these materials also show to have a negative magnetoresistance, as predicted by the simple argument of the previous section [26, 27]. In particular a negative magnetoresistance up to -30% was measured in TaAs samples [30] and up to -10% in doped NbP [31].

Chapter 3

Relativistic hydrodynamics

Contents

For the purpose of this thesis, we are interested in the study of hydrodynamic fluctuations, that are the small, long-wavelength fluctuations near thermal equilibrium. In this chapter we will mostly follow the explanation given by P. Kovtun in his review [32].

From now on we will work in a flat spacetime with metric $\eta_{\mu\nu} = \text{diag}(-1, 1, \ldots, 1)$, in d spatial dimensions and in natural units $c = k_B = \hbar = 1$,[1] later on we will specialize to the case $d = 3$.

[1] In this thesis we are going to apply the theory of relativistic hydrodynamics to a condensed matter system. In this system it is the Fermi velocity v_F to take the role of c in

3.1 Hydrodynamic variables and equations

Hydrodynamics (both classic and relativistic) is the theory that studies the conservation laws of everything that is conserved (usually energy, momentum and mass or particle number), with additional assumptions about the dissipative response and the thermodynamics of the system (such as the equation of state). In this first part we will only deal with normal hydrodynamics, so no superfluids (fluids with zero viscosity) or external electromagnetic fields.

We know from Noether's theorem that conserved quantities (charges) are due to some underlying symmetry of the system (e.g. momentum conservation arises from the symmetry under spatial translations): to each microscopic global continuous symmetry is associated a conserved current that implies a conserved charge. This gives us a continuity equation for each conserved charge of the system. In the case of non-relativistic hydrodynamics, for example, we have the conservation of mass (thus a continuity equation for the mass density $\rho(t, \mathbf{x})$), conservation of momentum (that contains terms proportional to the pressure $p(t, \mathbf{x})$), conservation of energy $\epsilon(t, \mathbf{x})$ (that, together with the two aforementioned equations, is linked to the conservation of entropy) and an equation of state [33]. There are therefore six fields ($\rho, \mathbf{v}, p$ and ϵ) and six equations, this completes our description of hydrodynamics.

In a relativistic theory the fundamental symmetries are: spacetime translations, rotations and boosts. There may be other conserved charges and we will focus on the simple case of a generic conserved current J^μ under a global $U(1)$ symmetry. The spacetime translations imply a conserved stress-energy tensor, but it is more convenient to define it by the variation of the action with respect to the metric (and then we will set the metric to be Minkowski metric, since we work in flat space) rather than by Noether's theorem. It does not matter that we do not know the action of the system, we will find the form of the tensor from other assumptions. With this definition $T^{\mu\nu}$ is automatically symmetric ($T^{\mu\nu} = T^{\nu\mu}$) and conserved [34]

$$T^{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}} \bigg|_{g_{\mu\nu}=\eta_{\mu\nu}}$$

where g is the metric determinant. The conservation law is simply

$$\partial_\mu T^{\mu\nu} = 0 \tag{3.1}$$

the Weyl equations, so for the application part in chapter 4 we will consider $v_F = 1$ as the effective speed of light.

The conserved currents related to rotations and boosts are given by the rank 3 tensor $\mathcal{M}^{\mu\nu\lambda} = x^\mu T^{\nu\lambda} - x^\nu T^{\mu\lambda}$, but these are always identically conserved $\partial_\lambda \mathcal{M}^{\mu\nu\lambda} = 0$ thanks to (3.1), so they do not imply any further conservation law. The only remaining symmetry is the $U(1)$ symmetry (we work with J^μ as a vector current, not an axial one) that gives us the conservation equation

$$\partial_\mu J^\mu = 0 \tag{3.2}$$

We will often think of these as the conservation of energy density for the time component of equation (3.1), of momentum density for the spatial components of (3.1) and of particle density (3.2).

In d spatial dimensions there are in total $d+1$ equations from (3.1) and one equation from (3.2), but $T^{\mu\nu}$ has $(d+1)(d+2)/2$ independent components (due to the fact that it is symmetric) while J^μ has $d+1$ components. In general there are not enough equations to solve for all the variables, but the hydrodynamic assumption comes in the fact that the system can be fully characterized by $d+2$ fields over spacetime: a local temperature $T(x)$, a chemical potential $\mu(x)$ and a velocity field $\mathbf{v}(x)$.

The idea behind this choice can be understood as follows: temperature and chemical potential are needed to define the local equilibrium state (by a procedure of coarse-graining, we can imagine to slice the system in cells that are microscopically large, therefore allowing a thermodynamic description, but macroscopically small enough to be identified by a single point in space). The velocity field instead gives a prescription on how to move from one of these small volume to the adjacent ones and allows us to have a global view of the system [35, 36].

The particular preference of using T and μ is arbitrary, but we will follow the usual convention to keep these as our hydrodynamic variables. However, instead of working with the spatial velocity $\mathbf{v}$, we will use the four-velocity vector u^μ, that carries the same information, but it is more suitable for relativistic calculations. u^μ is related to $\mathbf{v}$ by $u^\mu = \gamma(\mathbf{v})(1, \mathbf{v})$, with $\gamma(\mathbf{v})$ the Lorentz factor and it satisfies $u^2 = -1$.

Hydrodynamics deals with states that are only slightly off of thermal equilibrium, so we will describe these non-equilibrium state in terms of slowly varying fields $\mu(x)$, $T(x)$ and $u^\mu(x)$.

3.1.1 Hydrodynamic regime

Hydrodynamics is an effective theory, so it is important to specify its regime of validity: under what conditions can we expect hydrodynamics to give a

correct description of the system under consideration? We should in general keep in mind three different length scales (and respective time scales): l_{mfp} is the mean free path of the microscopic constituents, λ is the length scale over which $T^{\mu\nu}$ and J^{μ} vary, while L is the typical length scale of the whole system.

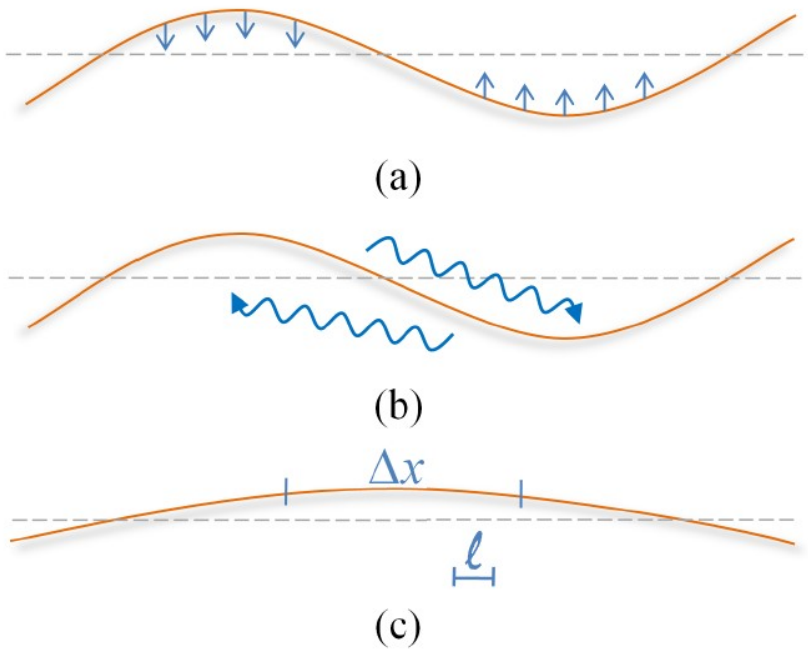

Figure 3.1: Image from [37]. Relaxation of the different types of excitations. The dashed line denote the global equilibrium values and the curved solid line some perturbed quantity. (a): perturbations for non-conserved quantities (the QP modes) can relax back to equilibrium quickly and locally, i.e. deviations separated at length scales larger than the relaxation length l relax independently, thanks to the frequent scattering events. (b): conserved quantities can only relax through transport, i.e. excesses have to be transported to regions with deficits to achieve equilibrium. (c): in a spacetime region with $l_{\text{mfp}} \ll \Delta x \ll \lambda$ and $\tau_{\text{mfp}} \ll \Delta t \ll t_{\lambda}$ a system can be considered in local thermodynamic equilibrium, specified by the local values of the conserved quantities.

When the constituents of the system are weakly coupled, the dynamic of the system can be described with the Fermi liquid theory or kinetic theory: in this regime the microscopic quasiparticle-like (QPs) degrees of freedom that constitute the system interact rarely with each other, thus the QP-QP scattering time is much longer then the QP-defect or QP-phonon scattering

time and the QPs are long-lived. This in turn implies that momentum and charge are conserved on scales that are much shorter than the mean free path, so $\lambda \ll l_{\text{mfp}}$. Hydrodynamics instead becomes sensible when the QPs are strongly coupled: now the QP-QP scattering time is very short and these degrees of freedom quickly relax at local thermodynamic equilibrium [38].

The hydrodynamic regime requires that λ is much larger than l_{mfp}, so that a local thermodynamic equilibrium is reached on short time scales. In this regime we can promote μ and T to be local fields $\mu(x)$ and $T(x)$ that vary on the scale λ. Hence for hydrodynamics to be a sensible theory we require that $l_{\text{mfp}} \ll \lambda \ll L$. In Fourier space this requirement is equivalent to asking that hydrodynamics only works at small vector numbers and frequency, so in general we will require that $\mathbf{k} \to 0$, $\omega \to 0$. More precisely, the requirement that the QPs are short-lived comes from the requirement that $\hbar\omega \ll k_B T$.

In a condensed matter system the usual requirement is that the length scale over which the thermodynamic variables vary is long compared to the microscopic mean free path, so that

$$\left|\frac{\partial_x F}{F}\right| \ll \frac{1}{l_{\text{mfp}}}$$

with F a generic hydrodynamic field (this inequality should not be taken too strictly, since F could be zero in certain points).

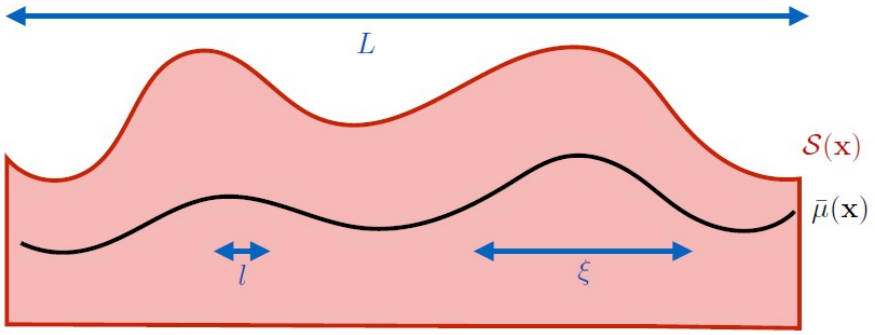

Figure 3.2: Image from [39]. The typical hydrodynamic regime, with the three length scales: L the size of the system, ξ the long wavelength perturbation to the hydrodynamic variables, l the microscopic scattering length that keeps the system in local thermodynamic equilibrium.

This requirement is also equivalent to asking that $\mu \ll T$, if instead $T \ll \mu$ Fermi liquid becomes the correct theory to use. We can intuitively understand this condition noting that Fermi liquid is a theory that describes

the dynamics of quasiparticles close to the Fermi surface, and this description works even at zero temperature. On the other hand hydrodynamics does not require particles, it is a sensible theory even at $\mu = 0$, but it requires a finite temperature, so $\mu \ll T$ is needed.

3.2 Constitutive relations

Given a timelike vector u^μ, it indicates the axis of proper time for each point of the four-space. Any generic vector F^μ gives rise to a scalar and a vector in the form [40]

$$f = -u_\mu F^\mu \tag{3.3a}$$
$$f^\mu = \Delta_\nu{}^\mu F^\nu \tag{3.3b}$$

With $\Delta^{\mu\nu} = \eta^{\mu\nu} + u^\mu u^\nu$ the projection operator [34] and $\eta_{\mu\nu}$ the metric tensor $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, \ldots, 1)$. The projector satisfies $u^\mu \Delta_\mu{}^\nu = 0$ and $\Delta^2 = \Delta$. The scalar is the projection of F^μ on the axis of proper time, while the vector is the projection of F^μ into proper space. With these new definitions, the following identity holds

$$F^\mu = f u^\mu + f^\mu$$

Generalizing this procedure (with particular attention to the symmetries of the tensor) we can decompose the stress-energy tensor and the current in the following form (this is the usual decomposition of a rank 2 tensor in $SO(3)$ irreducible terms)

$$T^{\mu\nu} = \mathcal{E} u^\mu u^\nu + \mathcal{P}\Delta^{\mu\nu} + (q^\mu u^\nu + q^\nu u^\mu) + t^{\mu\nu} \tag{3.4a}$$
$$J^\mu = \mathcal{N} u^\mu + j^\mu \tag{3.4b}$$

Where $\mathcal{E}$, $\mathcal{P}$ and $\mathcal{N}$ are scalars, the vectors q^μ and j^μ are transverse and the tensor $t^{\mu\nu}$ is transverse, symmetric and traceless. We can explicitly write these coefficients as we did in (3.3) and we find

$$\mathcal{E} = u_\mu u_\nu T^{\mu\nu} \qquad \mathcal{P} = \frac{1}{d}\Delta_{\mu\nu} T^{\mu\nu} \qquad \mathcal{N} = -u_\mu J^\mu \tag{3.5a}$$
$$q_\mu = -\Delta_{\mu\alpha} u_\beta T^{\alpha\beta} \qquad j_\mu = \Delta_{\mu\nu} J^\nu \tag{3.5b}$$
$$t_{\mu\nu} = \frac{1}{2}\big(\Delta_{\mu\alpha}\Delta_{\nu\beta} + \Delta_{\nu\alpha}\Delta_{\mu\beta} - \frac{2}{d}\Delta_{\mu\nu}\Delta_{\alpha\beta}\big) T^{\alpha\beta} \tag{3.5c}$$

The hydrodynamic assumption comes in the fact that we can express these coefficients in terms of the hydrodynamic variables. The expressions of

$T^{\mu\nu}$ and J^μ in terms of T, μ and u^μ are called constitutive relations.[2] For example, the scalar coefficients $\mathcal{E}$, $\mathcal{P}$ and $\mathcal{N}$ will be constructed by a particular combination of all the possible scalars we can make with the hydrodynamic variables: μ, T, but also $\partial_\mu u^\mu$ and higher-order terms.

Ideal hydrodynamics describes systems without dissipation. The core idea of hydrodynamics is to modify the equations of ideal hydrodynamics by adding dissipative terms that are proportional to the gradients of the hydrodynamic variables. In the hydrodynamic regime (long wavelength and small frequency) we assume that the gradients of the hydrodynamic variables are small, so that the hydrodynamic fields vary very slowly on the microscopic scale of the system. With this ansatz we can apply a procedure of gradient or derivative expansion: since we assumed that the gradients are small, we can create a list of terms made from the hydrodynamic variables (scalars, transverse vectors and tensors) and sort them depending on the number of derivatives. Ideal hydrodynamics will only contain terms that have no derivatives at all in the hydrodynamic variables (zeroth order), in principle we could then add first-order terms, second-order ones and so on and so forth.

We will limit ourselves to a first-order expansion: this is already complex enough and allows the description of a wide variety of effects, furthermore there are serious theoretical problems in the second-order expansion. At second-order hydrodynamics is not linear anymore, this allows for interactions between the hydrodynamic modes leading to a breakdown of the gradient expansion itself [32].

3.3 Ideal hydrodynamics

As we mentioned, ideal hydrodynamics corresponds to the zeroth order in the gradient expansion of the constitutive relations. This alone allows us to exclude q^μ, j^μ and $t^{\mu\nu}$: they are all transverse and thus they have to contain derivative terms.[3] The three remaining coefficients $\mathcal{E}$, $\mathcal{P}$ and $\mathcal{N}$ must be build out of μ and T alone for a similar reason.

Assuming isotropy of spacetime we can say that the stress-energy tensor of a system in static equilibrium must have the form $T^{\mu\nu} = \mathrm{diag}(\epsilon, p, \ldots, p)$ (with ϵ the equilibrium energy density and p the pressure). Following the

[2] In some texts constitutive relations identify only the dissipation terms of the stress-energy tensor and of the current.

[3] The only vector that can be made without derivatives is the velocity field u^μ, that is not transverse by definition.

same idea, J^μ has to be $J^\mu = (\rho, \mathbf{0})$ (ρ is the equilibrium charge density). If the fluid is moving at constant velocity u^μ we can perform a Lorentz boost with velocity $-u^\mu$ to find the corresponding energy momentum tensor and current observed by a stationary frame. This procedure gives us [34, 41]

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + p\Delta^{\mu\nu} = (\epsilon + p)u^\mu u^\nu + p\eta^{\mu\nu} \tag{3.6a}$$

$$J^\mu = \rho u^\mu \tag{3.6b}$$

For the zeroth order we claim that these are the form taken by the constitutive relations, when we promote ϵ, p, u^μ and ρ to slowly varying fields that depend on the hydrodynamic variables. Checking back the equations (3.4) we see that we can identify $\mathcal{E}(x) = \epsilon(x)$ (more correctly it should be $\epsilon(\mu(x), T(x))$), $\mathcal{P}(x) = p(x)$ and $\mathcal{N}(x) = \rho(x)$ that are respectively the local energy density, the local pressure and the local charge density. The (generally unknown) equilibrium equation of state of the system provides a relation $p(T, \mu)$. From this we can find the entropy density $s = \partial p/\partial T$, the charge density $\rho = \partial p/\partial \mu$ and the energy density by Euler equation $\epsilon = -p + Ts + \mu\rho$.

Applying (3.6) to the conservation equations (3.1) for the stress energy tensor and projecting along u^μ we can write[4]

$$\begin{aligned} &u_\nu \partial_\mu T^{\mu\nu} = 0 \\ &u_\nu [u^\nu \partial_\mu (\epsilon + p) u^\mu + (\epsilon + p) u^\mu \partial_\mu u^\nu] = -u^\mu \partial_\mu p \\ &\partial_\mu [(\epsilon + p) u^\mu] = u^\mu \partial_\mu p \end{aligned}$$

Using Euler equation $\epsilon + p = Ts + \mu\rho$, remembering that $\mathrm{d}p = s\,\mathrm{d}T + \rho\,\mathrm{d}\mu$ and using the conservation equation for the current, that in the ideal case reads $\partial_\mu(\rho u^\mu) = 0$ (3.2), we find

$$\partial_\mu (s u^\mu) = 0 \tag{3.7}$$

We identify the term between brackets as the entropy current and this equation describes the (exact) conservation of entropy in ideal hydrodynamics: entropy does not increase in the ideal case.[5] This feature is only true at the zeroth-order, adding dissipation terms will lead to (local) entropy production in general.

[4] Noting that $u_\mu \partial_\nu u^\mu = \frac{1}{2}\partial_\nu (u^\mu u_\mu) = 0$.

[5] Actually entropy can increase in the presence of discontinuity in the hydrodynamic flow (shock waves) even in the ideal case [33].

3.4 First-order hydrodynamics

From ideal hydrodynamics we now move to the first-order expansion. As we mentioned, the idea is to modify the equations of ideal hydrodynamics with terms that are linear in first derivative; usually the constitutive relations are written as

$$T^{\mu\nu} = (\epsilon + p)u^\mu u^\nu + p\eta^{\mu\nu} + \tau^{\mu\nu} \tag{3.8a}$$

$$J^\mu = \rho u^\mu + \nu^\mu \tag{3.8b}$$

with $\tau^{\mu\nu}$ and ν^μ the first-order dissipative correction to the ideal hydrodynamic equations.

3.4.1 Frame choice

When we move to the first-order expansion we encounter an ambiguity that we have to deal with. From a thermodynamic point of view the fields we use as our hydrodynamic variables are well defined at equilibrium, but do not have a unique definition out of equilibrium. This means that we are always free to redefine the fields with corrections that are first order in derivatives,[6] so we can have different definitions of $\mu(x)$, $T(x)$ and $u^\mu(x)$ that are in agreement only at equilibrium. We call frame a particular definition of the hydrodynamic fields, the transformations

$$\mu(x) \longrightarrow \mu'(x) = \mu(x) + \delta\mu(x) \tag{3.9a}$$

$$T(x) \longrightarrow T'(x) = T(x) + \delta T(x) \tag{3.9b}$$

$$u^\mu(x) \longrightarrow u'^\mu(x) = u^\mu(x) + \delta u^\mu(x) \tag{3.9c}$$

identify a certain frame transformation, allowing us to move from one definition to another. It should be noted that δu^μ has to be transverse, so that $u_\mu \delta u^\mu = u'_\mu \delta u^\mu = 0$, because the normalization condition $u^2 = -1$ must hold in the primed and unprimed frame.

As we said the hydrodynamic fields do not have a good microscopic definition out of equilibrium, but there are other quantities that are well defined outside of equilibrium, such as the stress-energy tensor $T^{\mu\nu}(x)$ and the current $J^\mu(x)$. The form of these observables has to be independent on the redefinition of the fields (3.9), this means that the hydrodynamic fields are simply auxiliary parameters used to parametrize $T^{\mu\nu}$ and J^μ.

[6] This is necessary, so that when the system approach the equilibrium the gradients go to zero and the various definitions give the same correct answer.

Following the same logic we used earlier to discuss the hydrodynamic fields, we know that the scalar coefficients in the constitutive relations (3.4) must have the form

$$\mathcal{E} = \epsilon(T, \mu) + f_{\mathcal{E}}(\partial\mu, \partial T, \partial u) \tag{3.10a}$$

$$\mathcal{P} = p(T, \mu) + f_{\mathcal{P}}(\partial\mu, \partial T, \partial u) \tag{3.10b}$$

$$\mathcal{N} = \rho(T, \mu) + f_{\mathcal{N}}(\partial\mu, \partial T, \partial u) \tag{3.10c}$$

with ϵ, p and ρ defined by the state of the system in equilibrium, while the f are first-order corrections that depend on the frame choice.

Using the frame transformation equations (3.9) and applying them to (3.5), while keeping in mind that the energy momentum tensor and the current do not change and using the transversality of the various terms, we arrive at the following results[7]

$$\delta\mathcal{E} = 0 \qquad \delta\mathcal{P} = 0 \qquad \delta\mathcal{N} = 0 \tag{3.11a}$$

$$\delta q_\mu = -(\mathcal{E} + \mathcal{P})\delta u_\mu \qquad \delta j_\mu = -\mathcal{N}\delta u_\mu \tag{3.11b}$$

$$\delta t_{\mu\nu} = 0 \tag{3.11c}$$

Equations (3.11b) tell us that the redefinition of the two vectors depends on the redefinition of the velocity field. We are therefore free to pick a certain velocity definition such that $j_\mu = 0$ (this choice is called Eckart frame [40]) or $q_\mu = 0$ (the Landau frame [33]). The first choice means to pick a velocity that follows the particles motion, so that there's no charge flow in the rest flame of the fluid, while the Landau's choice sets to zero the energy flow. The meaning of this freedom comes from the fact that we never actually defined what the velocity field was: in non relativistic hydrodynamics the velocity field describes the motion of the fluid particles, but in relativity there is a mass flux each time there is an energy flux, so we have to be careful defining the velocity field. Landau defines the velocity relating it to the energy flux, while Eckart link the velocity to the movement of the charges. From equations (3.11a) and (3.10) we notice that we can still use our freedom to define T and μ in such a way that two of the three functions $f_{\mathcal{E}}$, $f_{\mathcal{N}}$ and $f_{\mathcal{P}}$ go to zero; it is conventional to set $\mathcal{E} = \epsilon$ and $\mathcal{N} = \rho$.

[7] Just the first one: $\mathcal{E} = u_\mu u_\nu T^{\mu\nu}$. Since $T^{\mu\nu}$ does not change in a different frame, we have $\delta\mathcal{E} = 2\delta u_\mu u_\nu T^{\mu\nu}$, but $T^{\mu\nu}$ has transverse terms (that go to zero when multiplied by u_ν) and a single longitudinal term (that goes to zero when multiplied by δu_μ), so we find $\delta\mathcal{E} = 0$.

The Landau matching conditions that define the Landau frame are sometimes expressed in the following form

$$\tau^{\mu\nu}u_\nu = 0 \qquad \text{and} \qquad \nu^\mu u_\mu = 0 \tag{3.12}$$

with $\tau^{\mu\nu}$ and ν^μ the first order corrections to the constitutive equations.

It is possible to develop hydrodynamics in a frame-independent form, expressing $T^{\mu\nu}$ and J^μ only in terms of frame-independent quantities (see [42]), but for the purpose of this thesis we will not do so.

3.4.2 Constitutive relations: Landau frame

We will now specifically focus on the Landau frame, that is the one that we are going to use in the next chapter. In this frame, as a reminder, we have $\mathcal{E} = \epsilon$, $\mathcal{N} = \rho$ and $q^\mu = 0$. We still have to express $\mathcal{P}$, $t^{\mu\nu}$ and j^μ in terms of the hydrodynamic variables and their first derivatives. We now list all the possible scalars, transverse vectors and transverse traceless symmetric tensors that can be build from zeroth and first-order terms.

For the scalars, we certainly have T and μ at zeroth order, but we also have one-derivative terms: $u^\nu\partial_\nu T$, $u^\nu\partial_\nu\mu$ and $\partial_\mu u^\mu$. There are also three transverse vectors that we can make: $\Delta^{\mu\nu}\partial_\nu T$, $\Delta^{\mu\nu}\partial_\nu\mu$ and $\Delta^{\mu\nu}u^\lambda\partial_\lambda u_\nu$ and only one traceless transverse symmetric tensor[8]

$$\sigma^{\mu\nu} = \Delta^{\mu\alpha}\Delta^{\nu\beta}\Big(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{d}\eta_{\alpha\beta}\partial_\lambda u^\lambda\Big)$$

$\mathcal{P}$ is the only scalar left in the constitutive relations, we have to write it in terms of the possible scalars we listed above. We already showed that $\mathcal{P}$ has the form $\mathcal{P} = p(\mu, T) + f_{\mathcal{P}}(\partial\mu, \partial T, \partial u)$ (equation (3.10)), so we write

$$\mathcal{P} = p + c_1\, u^\nu\partial_\nu T + c_2\, u^\nu\partial_\nu\mu + c_3\partial_\mu u^\mu + O(\partial^2)$$

where the c_i (with $i = 1, 2, 3$) are some unknown coefficients and p is the pressure in the local rest frame of the fluid (that depends on T and μ). If we apply the zeroth-order constitutive equations (3.6) to the two scalar equations $u_\mu\partial_\nu T^{\mu\nu} = 0$ and $\partial_\mu J^\mu = 0$ we find two relations that link together the three one-derivative terms that appear in the expression for $\mathcal{P}$. The three first-order scalar terms are not independent, we are thus free to express $\mathcal{P}$ in terms of only one of these scalars: it is conventional to keep the term $\partial_\mu u^\mu$, giving us

$$\mathcal{P} = p - \zeta\partial_\mu u^\mu + O(\partial^2)$$

[8] We could in principle create a few other terms with the Levi-Civita symbol, but these terms violate parity and time-reversal invariance.

where ζ (that we can recognize as the bulk viscosity[9]) is a coefficient that remains undetermined in hydrodynamics and depends on the microscopic theory.

Proceeding in the same fashion, we had three transverse vectors we could create at first order in the derivative expansion, so j^μ would be a linear combination of them. Again, we also have one transverse vector equation $\Delta_{\mu\alpha}\partial_\beta T^{\alpha\beta} = 0$ that we can use to find a relation between the three allowed vectors, this gives us the opportunity to only keep two linearly independent vectors. It is again conventional to expand j^μ in the form

$$j^\mu = -\sigma T \Delta^{\mu\nu}\partial_\nu\left(\frac{\mu}{T}\right) + \chi_T \Delta^{\mu\nu}\partial_\nu T + O(\partial^2)$$

where we can identify σ as the charge conductivity, while χ_T we will see that has to be zero to preserve the positivity of entropy production. These coefficients must again be determined from the microscopic theory.

We only had one transverse traceless symmetric tensor, so the only possible expression for $t^{\mu\nu}$ is

$$t^{\mu\nu} = -\eta\sigma^{\mu\nu} + O(\partial^2)$$

with η the shear viscosity.

Putting all back together we find the constitutive relations in the Landau frame at first order. They are

$$\begin{aligned} T^{\mu\nu} &= (\epsilon + p)u^\mu u^\nu + p\eta^{\mu\nu} + \\ &\quad - \eta\Delta^{\mu\alpha}\Delta^{\nu\beta}\left(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{d}\eta_{\alpha\beta}\partial_\lambda u^\lambda\right) - \zeta\Delta^{\mu\nu}\partial_\lambda u^\lambda \end{aligned} \tag{3.13a}$$

$$J^\mu = \rho u^\mu - \sigma T\Delta^{\mu\nu}\partial_\nu\left(\frac{\mu}{T}\right) + \chi_T\Delta^{\mu\nu}\partial_\nu T \tag{3.13b}$$

The first assumption of hydrodynamics (that limits ourselves to the fields T, μ and u^μ), together with the assumptions of small fluctuations (that allows us to perform a gradient expansion) and Lorentz symmetry, restrict the expressions of the energy momentum tensor and current to just four transport coefficients (actually three, since as we said χ_T must be zero). These equations are supplemented by the equation of state that expresses the pressure in terms of the hydrodynamic variables $p(T,\mu)$ and from which we can find $\epsilon(T,\mu)$ and $\rho(T,\mu)$.

The coefficients η, σ, ζ and χ_T are called transport coefficients, they are parameters that are left undetermined by the theory of hydrodynamics and, like in any effective theory, they need to be determined by the microscopic

[9]Sometimes called second viscosity or volume viscosity.

theory, for example using the linear response theory, that allows to find a relation between the transport coefficients and the correlation functions of $T^{\mu\nu}$ and J^{μ}. Since their definitions come from the microscopic theory, the value of these coefficients is frame-independent: in another frame they can appear in a different position in the constitutive relations, but their value is fixed for a given system.

We will not do so, but it is straightforward to apply the same procedure to find the constitutive relations in the Eckart frame. Since the Eckart frame define the velocity field as the velocity of the matter it is better suited for a non-relativistic limit, allowing one to find the usual non-relativistic equations of classic hydrodynamics [33]; on the other hand, the form of the expressions make them not well suited for computations around the vacuum equilibrium state $\rho = 0$. For completeness we will write the form of the constitutive equations in the Eckart frame at first order

$$T^{\mu\nu} = \epsilon u^{\mu} u^{\nu} + p\Delta^{\mu\nu} + (q^{\mu} u^{\nu} + q^{\nu} u^{\mu}) - \eta\sigma^{\mu\nu} - \zeta\Delta^{\mu\nu}\partial_{\lambda} u^{\lambda} \tag{3.14a}$$

$$J^{\mu} = \rho u^{\mu} \tag{3.14b}$$

where $q^{\mu} = (\sigma T \Delta^{\mu\nu}\partial_{\nu}(\mu/T) - \chi_T \Delta^{\mu\nu}\partial_{\nu} T)(\epsilon + p)/\rho$.

3.5 Entropy current

In the case of ideal hydrodynamics we defined the entropy current as $S^{\mu} = su^{\mu}$ and we showed that this is exactly conserved at the zeroth order $\partial_{\mu} S^{\mu} = 0$. We now claim that there is a first-order correction (with respect to the hydrodynamic variables) to the entropy current $S^{\mu} = su^{\mu} +$(first-order terms) which satisfies

$$\partial_{\mu} S^{\mu} \geq 0$$

when the equations of hydrodynamics are fulfilled and that is exactly conserved in an equilibrium state. In general there is not a unique expression for the entropy current, but we will proceed by analogy, trying to write thermodynamic relations in a generic covariant form.

We start from the Euler relation of classic thermodynamics $Ts = p + \epsilon - \mu\rho$. The covariant form of this expression is [43]

$$TS^{\mu} = pu^{\mu} - T^{\mu\nu} u_{\nu} - \mu J^{\mu} \tag{3.15}$$

Using equations (3.4) we can write the entropy current in terms of the

hydrodynamic coefficients and vectors

$$S^\mu = \left[s + \frac{1}{T}(\mathcal{E} - \epsilon) - \frac{\mu}{T}(\mathcal{N} - \rho) \right] u^\mu + \frac{1}{T} q^\mu - \frac{\mu}{T} j^\mu$$

This expression is frame-independent, but we can specialize it to the Landau and Eckart frame, which simplifies it a lot. If we now impose that the gradient of the entropy current has to be (locally) positive, with the help of thermodynamic relations we find some constraints on the values of the transport coefficients. Specifically the constraints are[10]

$$\eta \geq 0 \qquad \zeta \geq 0 \qquad \sigma \geq 0 \qquad \chi_T = 0$$

We show an example of this procedure here: in the Landau frame the entropy current reads

$$S^\mu = s u^\mu - \frac{\mu}{T} \nu^\mu$$

Calculating the divergence of this expression, remembering Euler equation $\epsilon + p = sT + \mu\rho$, using $\mathrm{d}p = \rho\, \mathrm{d}\mu + s\, \mathrm{d}T$ and using the conservation equations for the stress-energy tensor (projected along u_ν) and the current we obtain

$$\partial_\mu \Big(s u^\mu - \frac{\mu}{T} \nu^\mu \Big) = -\nu^\mu \partial_\mu \Big(\frac{\mu}{T} \Big) - \frac{\tau^{\mu\nu}}{T} \partial_\mu u_\nu \geq 0$$

Where ν^μ is the dissipative first-order term of the current, while $\tau^{\mu\nu}$ is the first-order part of the stress-energy tensor. If the fluid is moving uniformly we can ignore the last term; inserting ν^μ in the equation gives

$$\sigma T \Delta^{\mu\nu} \partial_\nu \Big(\frac{\mu}{T} \Big) \partial_\mu \Big(\frac{\mu}{T} \Big) - \chi_T \Delta^{\mu\nu} \partial_\nu T \partial_\mu \Big(\frac{\mu}{T} \Big) \geq 0$$

This expression must be always positive (or zero) for all possible choice of μ, T and u^μ, this implies that the two terms must be independently positive. The first term is σT times a term that is always positive (it is a square), this implies $\sigma \geq 0$; the second term on the other hand is (always) non-negative only if $\chi_T = 0$.

Under these conditions the constitutive relations in the Landau frame are

$$\begin{aligned} T^{\mu\nu} &= (\epsilon + p) u^\mu u^\nu + p \eta^{\mu\nu} + \\ &\quad - \eta \Delta^{\mu\alpha} \Delta^{\nu\beta} \left(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{d} \eta_{\alpha\beta} \partial_\lambda u^\lambda \right) - \zeta \Delta^{\mu\nu} \partial_\lambda u^\lambda \end{aligned} \tag{3.16a}$$

$$J^\mu = \rho u^\mu - \sigma T \Delta^{\mu\nu} \partial_\nu \left(\frac{\mu}{T} \right) \tag{3.16b}$$

[10] It is possible to find the same result in linear response theory, without defining the entropy current. In this case $\chi_T = 0$ comes from the requirement that the theory must be invariant under the time-reversal symmetry.

3.6 Magnetohydrodynamic

In this thesis we will compute the thermoelectric transport matrix in the presence of a constant magnetic field, but to do so we need to also consider (non-dynamic) electromagnetic effects in our theory of hydrodynamics.[11]

The procedure to fix the constitutive relations is the same as before, but this time we have another field, the electromagnetic tensor $F^{\mu\nu}$,[12] which we can use to construct scalars and vectors: even with this new freedom, the constitutive relation for the stress-energy tensor remains the same in the Landau frame, but the current gains a new term proportional to $F^{\mu\nu}$ [13] [32, 45]

$$J^\mu = \rho u^\mu - \sigma_E T \Delta^{\mu\nu} \partial_\nu \left(\frac{\mu}{T}\right) + \sigma_E F^{\mu\nu} u_\nu \tag{3.17}$$

where σ_E is usually called quantum critical conductivity [3, 46]. Since none of the fields that appear in (3.17) carry charge, we can keep the normal partial derivative instead of the covariant one.

It is important to remember that the electromagnetic field produces new constraints on the hydrodynamic regime, in particular it requires that $E, B \ll T^2$ for hydrodynamics to be a sensible theory. This means that the electric and magnetic fields cannot be too strong, otherwise secondary degrees of freedom come into play (in particular the Landau levels of the system) [44]. The new scale introduced by the electromagnetic field is parallel to the scale $\mu \leq T$ already imposed by hydrodynamics itself, the relation between these scaling is not clear and well defined to date.

The second correction to the equations of hydrodynamics comes from standard field theory: in presence of an external electromagnetic field the conservation equations needs to be modified, since the current couples to the electromagnetic field, leading to

$$\partial_\mu T^{\mu\nu} = F^{\nu\lambda} J_\lambda \tag{3.18a}$$

$$\partial_\mu J^\mu = 0 \tag{3.18b}$$

Energy and momentum are not conserved anymore, because the external

[11] In the most general sense magnetohydrodynamic becomes rather complicated, see [44].

[12] Defined as usual as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, with A_μ the electromagnetic four-potential.

[13] Heuristically we can argue that $F^{\mu\nu}$ is of order one in derivative expansion, in fact it is the derivative of the field A^μ that is of order zero (the electric potential is a form of electrochemical potential μ). It should be noted, however, that this is not necessarily always the case and that the derivative expansion in the presence of $F^{\mu\nu}$ is still an open problem.

field does work on the system. These equations will gain extra terms due to anomalies and impurities later on.

3.7 Linear response theory

3.7.1 Introduction

In this section we will quickly resume the main idea behind linear response theory, that is to describe the response of a system at equilibrium to a small perturbation. We consider a set of hydrodynamic fields $\phi_a(t, \mathbf{x})$ (that are microscopically well defined) and we couple them to their (weak slowly-varying) sources $\lambda_a(t, \mathbf{x})$. The sources are turned on at $t = -\infty$ and are slowly varying, at $t = 0$ they are turned off and the system is let free to evolve. In linearized hydrodynamics the fields ϕ (we drop the indices where not needed) are described by a set of linear equations that, in momentum space, reads[14]

$$\partial_t \phi_a(t, \mathbf{k}) + M_{ab}(\mathbf{k})\phi_b(t, \mathbf{k}) = 0$$

where M_{ab} is a matrix that comes from the conservation equations of hydrodynamics. This set of equations is valid in the hydrodynamic regime (long wavelength, small frequency: $\omega(\mathbf{k}) \to 0$ and $\mathbf{k} \to 0$) and describes the free evolution of the hydrodynamic fields at $t > 0$. We perform a Laplace transform in time

$$\phi(z, \mathbf{k}) = \int_0^\infty \mathrm{d}t \; e^{izt}\phi(t, \mathbf{k})$$

with z that has to lie on the upper half of the complex plane for the integral to converge. Transforming the set of equation we find

$$(-iz\delta_{ab} + M_{ab})\phi_b(z, \mathbf{k}) = \phi_a^0(\mathbf{k})$$

where we defined $\phi^0(\mathbf{k}) = \phi(t = 0, \mathbf{k})$, the value of the field just when the source is turned off. The values at $t = 0$ of the hydrodynamic variables are related to the values of the sources at the initial moment, that is just before we turn the sources off we have a relation that is local (wave number dependent) in space. For small fluctuations we can write

$$\phi_a^0(\mathbf{k} \to 0) = \chi_{ab}\lambda_b^0(\mathbf{k} \to 0) \qquad \Longrightarrow \qquad \chi_{ab} = \left(\frac{\partial \phi_a}{\partial \lambda_b}\right) \tag{3.19}$$

[14] Here the fields ϕ whose dynamics is described by linearized hydrodynamics are actually $\delta\phi$, that is the difference with respect to the equilibrium $\delta\phi(t, \mathbf{x}) = \phi(t, \mathbf{x}) - \phi$.

χ_{ab} is called static susceptibility, it is not a dynamical response quantity, but rather a static thermodynamic quantity that depends on the system. The small-$\mathbf{k}$ limit is to ensure the long wavelength validity of the hydrodynamic regime. With this definition the hydrodynamic equations can be formally solved in terms of the initial conditions

$$\phi_a(z, \mathbf{k}) = (K^{-1})_{ab}\chi_{bc}\lambda_c^0(\mathbf{k}) \tag{3.20}$$

with $K_{ab} = -iz\delta_{ab} + M_{ab}(\mathbf{k})$.

We now try to relate the expectation values of the hydrodynamic fields to the correlation functions of the system. We imagine to perturb the fields with slowly-varying (with respect to both $\mathbf{x}$ and t) sources that are adiabatically turned on at $t = -\infty$ and then turned off at $t = 0$, for example with $\lambda(t, \mathbf{x}) = e^{\varepsilon t}\lambda^0(\mathbf{x})\theta(-t)$ ($\varepsilon > 0$ is a small adiabatic parameter and $\theta(t)$ is the step function); we can then link the evolution of the system at $t > 0$ to the result we found in equation (3.20). To do so we perturb the Hamiltonian with a term that couples the hydrodynamic fields to their sources

$$\delta H(t) = -\int \mathrm{d}^d\mathbf{x}\ \lambda_a(t, \mathbf{x})\phi_a(t, \mathbf{x})$$

From first-order time-dependent perturbation theory in Quantum Mechanics we can find the perturbation of an operator $\phi(t, \mathbf{x})$ in the Heisenberg picture. If the system has a time-independent Hamiltonian and we add the small perturbation above the expected value of the observable changes as[15]

$$\delta\langle\phi_a(t, \mathbf{x})\rangle = -i\int_{-\infty}^{t} \mathrm{d}t'\langle[\phi_a(t, \mathbf{x}), \delta H(t')]\rangle$$

where the $\langle\dots\rangle$ means a thermal average $\langle\mathcal{O}\rangle = \mathrm{Tr}(\rho\mathcal{O})$ with ρ the density matrix (we work in the grand canonical ensemble, so the Heisenberg operators are defined with the Hamiltonian $H' = H - \mu N$, with the chemical potential μ as the Lagrange multiplier of the charge number operator N). Plugging together the two equations we find

$$\delta\langle\phi_a(t, \mathbf{x})\rangle = -\int_{-\infty}^{\infty} \mathrm{d}t' \int \mathrm{d}^d x'\ G^R_{ab}(t - t', \mathbf{x} - \mathbf{x}')\lambda_b(t, \mathbf{x}) \tag{3.21}$$

where G^R_{ab} is called retarded response function and it is defined as

$$G^R_{ab}(t - t', \mathbf{x} - \mathbf{x}') = -i\theta(t - t')\langle[\phi_a(t, \mathbf{x}), \phi_b(t', \mathbf{x}')]\rangle \tag{3.22}$$

[15] Many texts here do not have the minus sign in front, because they define the perturbation to the Hamiltonian $\delta H(t)$ with the plus sign.

G^R is also related to the Green or two-point function of the quantum system. It depends only on the differences of time and position, this is due to its invariance under spacetime translations. In Fourier space the expected value simplifies and the convolution gives

$$\delta\langle\phi_a(\omega,\mathbf{k})\rangle = -G^R_{ab}(\omega,\mathbf{k})\lambda_b(\omega,\mathbf{k}) \tag{3.23}$$

We still have to Laplace-Fourier transform equation (3.21) to link this result with the one obtained from linearized hydrodynamics. We start off by performing the Fourier transform in space, with our assumption on the form of the sources we find

$$\langle\phi_a(t,\mathbf{k})\rangle = -\int_{-\infty}^{0} \mathrm{d}t' e^{\varepsilon t'} G^R_{ab}(t-t',\mathbf{k})\lambda^0_b(\mathbf{k}) \tag{3.24}$$

We now Fourier transform only the retarded function in time

$$G^R(t-t',\mathbf{k}) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} G^R(\omega,\mathbf{k}) e^{-i\omega(t-t')}$$

Because of the step function in the definition of $G^R(t,\mathbf{k})$, this function is identically zero for $t<0$, this in turn means that $G^R(\omega,\mathbf{k})$ is an analytic function in the upper half-plane of complex ω and we are free to analytically continue $G^R(\omega,\mathbf{k})$ to the whole plane. After performing the t' integral we are left with

$$\langle\phi_a(t,\mathbf{k})\rangle = -\lambda^0_b(\mathbf{k}) \int \frac{\mathrm{d}\omega}{2\pi} G^R_{ab}(\omega,\mathbf{k}) \frac{e^{-i\omega t}}{i\omega+\varepsilon}$$

We can now multiply both sides by e^{izt} (with the prescription that $\mathrm{Im}(z)>0$ for convergence) and integrate over t from 0 to ∞ (that is, we perform the same Laplace transform we applied earlier to our linearized hydrodynamic equations)

$$\langle\phi_a(z,\mathbf{k})\rangle = -\lambda^0_b(\mathbf{k}) \int \frac{\mathrm{d}\omega}{2\pi} \frac{G^R_{ab}(\omega,\mathbf{k})}{(i\omega+\varepsilon)(i(\omega-z)+\varepsilon)}$$

As we mentioned, $G^R(\omega)$ is analytic in the upper half-plane, this allows us to perform the integral by closing the contour with $\mathrm{Im}(\omega)>0$. There are two poles inside the region, one at $\omega=i\varepsilon$ and one at $\omega=z+i\varepsilon$. The residue theorem gives us

$$\langle\phi_a(z,\mathbf{k})\rangle = -\lambda^0_b(\mathbf{k}) \frac{G^R_{ab}(z,\mathbf{k}) - G^R_{ab}(z=0,\mathbf{k})}{iz}$$

where the argument $z = 0$ is meant to be taken slightly above the real axis. We only need to find what $G^R(z = 0, \mathbf{k})$ is, to do so we look at equation (3.24) evaluated at $t = 0$. The argument of the integral is the Laplace transform evaluated at $z = 0$ (it is again meant to be slightly above the real axis, from $z = i\varepsilon$).

$$\langle \phi_a(t = 0, \mathbf{k}) \rangle = -\int_0^\infty \mathrm{d}t' \; e^{-\varepsilon t'} G^R_{ab}(t', \mathbf{k}) \lambda^0_b(\mathbf{k}) = -G^R_{ab}(z = 0, \mathbf{k}) \lambda^0_b(\mathbf{k})$$

From this equation we find that, in the long wavelength limit, $G^R(z = 0, \mathbf{k})$ is minus the static susceptibility $\chi(\mathbf{k})$ defined above (3.19), $G^R(z = 0, \mathbf{k}) = -\chi(\mathbf{k})$.

We can finally compare this result with our earlier result from linearized hydrodynamics obtaining (after renaming saturated indices)

$$-\frac{1}{iz}\big(G^R_{ab}(z, \mathbf{k}) + \chi_{ab}(\mathbf{k})\big)\lambda^0_b(\mathbf{k}) = (K^{-1})_{ac}\chi_{cb}\lambda^0_b(\mathbf{k}) \tag{3.25}$$

and we find an expression for the retarded response function matrix

$$G^R_{ab}(z, \mathbf{k}) = -(\delta_{ac} + iz(K^{-1})_{ac})\chi_{cb} \tag{3.26}$$

This function is always analytic in the upper half-plane of complex z, hence we can define $G^R(\omega, \mathbf{k})$ in the whole complex plane as the analytical continuation of $G^R(z, \mathbf{k})$ from the upper half-plane.

G^R has many properties, in particular it possible to show that

$$-\operatorname{Im} G^R_{aa}(\omega, \mathbf{k}) \geq 0 \qquad \text{for} \quad \omega \geq 0$$

This result, when applied to the hydrodynamic response functions, implies that the transport coefficients are all non-negative: $\sigma \geq 0$, $\zeta \geq 0$ and $\eta \geq 0$, without the need to define an entropy current.

The retarded functions have to be consistent with the symmetries of the theory, in particular time-reversal turns out to impose powerful constraints on the transport coefficients. Consider a Hermitian field $\phi_a(t, \mathbf{x})$ that transform under time reversal as $\Theta\phi_a(t, \mathbf{x})\Theta^{-1} = \eta_a\phi_a(-t, \mathbf{x})$, where Θ is the anti-unitary time-reversal operator and $\eta_a = \pm 1$ is the eigenvalue of ϕ_a. If the microscopic system is time-reversal invariant, i.e. $[H, \Theta] = 0$, then the retarded response functions must obey

$$G^R_{ab}(t, \mathbf{x}) = G^R_{ba}(t, -\mathbf{x})\eta_a\eta_b$$

On the other hand, if time-reversal is not a symmetry of the microscopic system, for example if there is an external magnetic field B, the Hamiltonian

satisfies $\Theta H(B)\Theta^{-1} = H(-B)$. From this assumptions the retarded functions have to satisfy the equation

$$G^R_{ab}(\omega, \mathbf{k}; B) = \eta_a \eta_b G^R_{ba}(\omega, -\mathbf{k}; -B) \tag{3.27}$$

This equation is the basis for the so called Onsager reciprocal relations [47]. This condition is not automatically satisfied in linearized hydrodynamics, instead it should be interpreted as a constraint on the form of the constitutive equations: in the limit $\mathbf{k} \to 0$, at $\omega = 0$, this condition applied to the hydrodynamic first-order equations implies that $\chi_T = 0$. Again, this is another way to constraint the transport coefficients without recurring to the entropy current.

3.7.2 Thermoelectric conductivity matrix

In this work we are interested in finding the thermoelectric transport coefficients in a linear response theory framework. Usually the charge current couples with the electric field, while the heat current couples with a temperature gradient, but there are also thermoelectric effects where the sources create mixed effects described by the full thermoelectric matrix. The relation is usually written as [35]

$$\begin{pmatrix} J_i \\ Q_i \end{pmatrix} = \begin{pmatrix} \Sigma_{ij} & \alpha_{ij} \\ T\bar{\alpha}_{ij} & \bar{\kappa}_{ij} \end{pmatrix} \begin{pmatrix} E_j \\ -\nabla_j T \end{pmatrix} \tag{3.28}$$

Where Σ is the electrical conductivity tensor, $\bar{\kappa}$ is the thermal conductivity tensor, α and $\bar{\alpha}$ are the thermoelectric tensors. In principle they could be different, but Onsager relations will imply certain symmetries on these coefficients. The σ_E that appears in the constitutive relations is not to be confused with Σ: σ_E cannot be determined by hydrodynamics and it depends on the microscopic theory, on the other hand Σ describes the macroscopic dynamics of the current with respect to an external electric field and will contain a contribution due to σ_E.

It should be noted that $\bar{\kappa}$ is not the usual thermal conductivity matrix measured in experiments: in experiments the boundary condition for the thermal conductivity is usually set by $\mathbf{J} = 0$, while in the above definition the boundary condition is $\mathbf{E} = 0$. The relation between κ_{ij} measured in experiments and $\bar{\kappa}_{ij}$ from the above definition is given by

$$\kappa_{ij} = \bar{\kappa}_{ij} - T\bar{\alpha}_{ik}\Sigma^{-1}_{kl}\alpha_{lj}$$

In this form the transport coefficients have to obey Onsager relations, this implies that the matrices α_{ij} and $\bar{\alpha}_{ij}$ are related by $\alpha_{ij}(B) = \bar{\alpha}_{ij}(B)$, with B an external magnetic field that breaks time-reversal invariance [35].

In linear response theory we would like to perturb the system with a small electric field and gradient temperature, from there, find how the electric and heat current react upon these perturbations. To do so, we have to first figure out the correct sources: for example the electric current does not couple to the electric field in the Hamiltonian, but instead it couples to the four-potential A^μ.

A disturbance in temperature couples to the Hamiltonian, and since we are working in the grand canonical ensemble, for a constant disturbance δT we find

$$H \to H - \frac{\delta T}{T}(H - \mu N)$$

For non-constant sources, and remembering that the the four-potential couples to the electric current, we have (note that μ is the equilibrium value)

$$\delta H = -\int \mathrm{d}^d x \left(\frac{\delta T(t,\mathbf{x})}{T}(\epsilon(t,\mathbf{x}) - \mu\rho(t,\mathbf{x})) + \delta A^\mu(t,\mathbf{x}) J_\mu(t,\mathbf{x}) \right)$$

We can fix the gauge by choosing the electric field to be minus the gradient of A^0, $E^i = -\partial_i A^0$, so we can ignore the $\delta A^i J_i$ terms and focus on $\delta A^0 \rho$

$$\delta H = -\int \mathrm{d}^d x \left(\frac{\delta T(t,\mathbf{x})}{T}(\epsilon(t,\mathbf{x}) - \mu\rho(t,\mathbf{x})) + \delta A^0(t,\mathbf{x}) \rho(t,\mathbf{x}) \right)$$

In classic thermodynamics the heat current is defined as [35] $\mathbf{J}_Q = T\mathbf{J}_S = \mathbf{J}_U - \mu\mathbf{J}_N$ where $\mathbf{J}_S$ is the entropy current while $\mathbf{J}_U$ and $\mathbf{J}_N$ are respectively the energy and charge current. We now claim that in the relativistic case $\epsilon - \mu\rho$ is the time component of the heat current four-vector (again, with μ as the equilibrium value). Hence the relativistic generalization to the heat current, in linear response theory, is simply

$$Q^i = T^{0i} - \mu J^i \tag{3.29}$$

Now that we identified the sources, we note that the conservation equations for the current and energy can be written as[16]

$$\begin{array}{l} \partial_t \rho + \partial_i J^i = 0 \\ \partial_t \epsilon + \partial_i T^{0i} = 0 \end{array} \quad \Longrightarrow \quad \begin{array}{l} \partial_t \rho = -\partial_i J^i \\ \partial_t(\epsilon - \mu\rho) = -\partial_i(T^{0i} - \mu J^i) = -\partial_i Q^i \end{array}$$

[16] We will always work without electric field at equilibrium, so the term $F^{\mu\nu}J_\nu$ that is on the right side of the stress-energy tensor equation does not appear.

As a last step, we recall from the previous section that, after a Laplace transform in time and a Fourier transform in space, any quantity F obeys in linear response (there are also other terms that depends on other sources we are not interested in indicated by the dots)

$$F(\omega,\mathbf{k}) = -\frac{G^R_{F;\epsilon-\mu\rho}(\omega,\mathbf{k}) - G^R_{F;\epsilon-\mu\rho}(0,\mathbf{k})}{i\omega}\frac{\delta T^0(\mathbf{k})}{T} + \\ - \frac{G^R_{F;\rho}(\omega,\mathbf{k}) - G^R_{F;\rho}(0,\mathbf{k})}{i\omega}\delta A^0(\mathbf{k}) + \dots$$

From this equation, using the conservation laws for the electric and heat current written above (once transformed in Fourier space), we obtain

$$F(\omega,\mathbf{k}) = \frac{1}{i\omega}\Big[-\frac{G^R_{F;\mathbf{Q}}(\omega,\mathbf{k}) - G^R_{F;\mathbf{Q}}(0,\mathbf{k})}{i\omega}\Big(-\frac{i\mathbf{k}\delta T^0(\mathbf{k})}{T}\Big) + \\ - \frac{G^R_{F;\mathbf{J}}(\omega,\mathbf{k}) - G^R_{F;\mathbf{J}}(0,\mathbf{k})}{i\omega}\delta\mathbf{E}^0(\mathbf{k})\Big] + \dots \quad (3.30)$$

When $F = \mathbf{Q}$ or $\mathbf{J}$ and $\mathbf{k} \to 0$ we recognize the coefficients of the electric field and and temperature gradient as $1/i\omega$ times the Kubo formulae for the thermoelectric transport coefficients $\bar{\kappa}$, α, $\bar{\alpha}$ and Σ, where we define

$$\Sigma_{ij}(\omega) = -\frac{G^R_{J_iJ_j}(\omega) - G^R_{J_iJ_j}(0)}{i\omega} \quad (3.31a)$$

$$\alpha_{ij}(\omega) = -\frac{G^R_{J_iQ_j}(\omega) - G^R_{J_iQ_j}(0)}{i\omega T} \quad (3.31b)$$

$$\bar{\alpha}_{ij}(\omega) = -\frac{G^R_{Q_iJ_j}(\omega) - G^R_{Q_iJ_j}(0)}{i\omega T} \quad (3.31c)$$

$$\bar{\kappa}_{ij}(\omega) = -\frac{G^R_{Q_iQ_j}(\omega) - G^R_{Q_iQ_j}(0)}{i\omega T} \quad (3.31d)$$

(usually the susceptibilities $G^R(0)$ are zero because of gauge invariance) [48].

3.7.3 Onsager reciprocal relations

We already talked about Onsager relations in the above Introduction to the Linear response theory and we quickly showed how the retarded response functions are related because of the time-reversal symmetry, from a quantum mechanical approach. Here we will review the full classical derivation from [35, 47], giving a bit more context about what this implies in terms of the thermoelectric transport matrix.

Consider the entropy of some closed system $S(x_i)$ as a function of fluctuating quantities x_i (considered as small variations from the equilibrium,

with the mean value $\bar{x}_i$ already subtracted so that $\bar{x}_i = 0$) and we write the probability distribution as $\varrho\, \mathrm{d}x_1\, \mathrm{d}x_2 \ldots \mathrm{d}x_n$ with the probability density function given by

$$\varrho \propto e^S \qquad \text{where} \qquad S = S_0 - \frac{1}{2}\beta_{ij} x_i x_j$$

where we expanded S in power series around the equilibrium solution. S_0 is the equilibrium entropy, the first derivative is zero (because the equilibrium is a maximum for S) and β_{ij} is a symmetric positive-definite matrix that comes from the second derivative of the entropy at equilibrium.

The constant in front of e^S can be found from the normalization condition

$$\int \varrho\, \mathrm{d}x_1\, \mathrm{d}x_2 \ldots \mathrm{d}x_n = 1$$

and from this we find

$$\varrho = \frac{\sqrt{\beta}}{(2\pi)^{n/2}} \exp\left(-\frac{1}{2}\beta_{ij} x_i x_j\right)$$

where β is the determinant of β_{ij}. Notice that ϱ is a Gaussian distribution.

We now define the quantities

$$X_i = -\frac{\partial S}{\partial x_i} = \beta_{ij} x_j$$

as the thermodynamic conjugate to the x_i (for example x_i could be a pressure fluctuation, so that X_i is the volume).[17]

We can now determine the mean value of the product

$$\langle x_i X_j \rangle = \delta_{ij} \tag{3.32}$$

and also, substituting this result in the definition of the thermodynamic conjugate, we can evaluate

$$\langle x_i x_j \rangle = \beta_{ij}^{-1} \qquad \text{and} \qquad \langle X_i X_j \rangle = \beta_{ij}$$

where β_{ij}^{-1} is an inverse matrix element.

[17] Note that, because of the linearity of the definition, the relation is reciprocal, i.e. if the entropy is expressed as a function of X_i, then

$$x_i = -\frac{\partial S}{\partial X_i}$$

We also define the correlation functions for the fluctuations of the x_i quantities as

$$\phi_{ij}(t'-t) = \langle x_i(t')x_j(t)\rangle \qquad \text{or simply} \qquad \phi_{ij}(t) = \langle x_i(t)x_j(0)\rangle$$

From the definition alone it is clear that $\phi_{ij}(t) = \phi_{ji}(-t)$, but there is another symmetry that arises from the symmetry under time reversal of the microscopic dynamics of the system. This second symmetry implies that it does not matter the order in which we select the fluctuating quantities, so $\langle x_i(t')x_j(t)\rangle = \langle x_i(t)x_j(t')\rangle$, i.e. $\phi_{ij}(t) = \phi_{ij}(-t)$. These two symmetries together also imply $\phi_{ij}(t) = \phi_{ji}(t)$.

If the fluctuations are quasi-stationary, i.e. the set of values $x_1, \ldots, x_n$ determines a state of partial equilibrium, then we can assume that the x_i vary with time and that the rate of change of the x_i is determined by the state itself, so that $\dot{x}_i = \dot{x}_i(x_1, x_2, \ldots, x_n)$. If the fluctuations are small, then we can expand in powers of the x_i to first order

$$\dot{x}_i = -\lambda_{ij}x_j$$

with constant coefficients λ_{ij}.

We rewrite this last equation expressing the RHS in terms of the thermodynamic conjugate $X_i = \beta_{ij}x_j$ and we obtain

$$\dot{x}_i = -\gamma_{ij}X_j \qquad \text{where} \qquad \gamma_{ij} = \lambda_{ik}\beta^{-1}_{kj}$$

The γ_{ij} are called kinetic coefficients.

Consider now the product average of two fluctuations and its symmetry under time reversal, as discussed earlier

$$\langle x_i(t)x_k(0)\rangle = \langle x_i(0)x_k(t)\rangle$$

We can take the derivative of this expression with respect to t and substitute the above equation for $\dot{x}_i$. We find

$$\gamma_{ij}\langle X_j x_k\rangle = \gamma_{kj}\langle x_i X_j\rangle$$

We showed earlier in (3.32) that $\langle X_i x_j\rangle = \delta_{ij}$, hence we obtain the Onsager relations

$$\gamma_{ik} = \gamma_{ki} \tag{3.33}$$

This derivation was done in a system without applied magnetic field B (along some direction) and assuming that the macroscopic variables x_i are

unaffected by time reversal. If a magnetic field is applied to the system then the Onsager relations become

$$\gamma_{ij}(B) = \gamma_{ji}(-B)$$

Now consider the case in which only one of the quantities x_i changes sign under time reversal, while the other remains the same. Then the relation is

$$\gamma_{ij} = -\gamma_{ji}$$

The above discussion is very general, but when the x_i are extensive local parameters then X_i are called affinities and their role is that of a generalized force, i.e. they force the system towards equilibrium. We usually also call $\dot{x}_i$ flux, that is the quantity that is affected by the presence of a non-zero affinity. In the thermoelectric transport framework the fluxes are the (components) of the currents, while the affinities are the gradients and the fields that drive the transport.

Onsager relations thus link together the response of certain quantities (the fluxes) of the system to the sources (the affinities), claiming that these responses are related by the aforementioned relations. This result implies, on the thermoelectric matrix, that each of the four matrices has to obey

$$\Sigma_{ij}(B) = \Sigma_{ji}(-B)$$

and similar relations for the other three matrices. Furthermore applying Onsager relation to the full thermoelectric matrix

$$\bar{\alpha}_{ij}(B) = \alpha_{ji}(-B) = \alpha_{ij}(B)$$

where in the last step we used the relation that the single matrices must obey.

Chapter 4

Weyl hydrodynamic

Contents

4.1 Anomalous hydrodynamics

The most important feature of chiral fluids is the fact that, even if they are macroscopic in nature, they inherit effects due to the quantum anomalies of the microscopic theory (a classical symmetry is broken by a chiral anomaly) [3]. Famous examples of this are the chiral magnetic effect and the chiral vortical effect. In particular we will focus on the effects of a quantum triangle anomaly $U(1)^3$.

In our hydrodynamic discussion we will indeed consider an axial current that is not conserved because of the anomaly. The hydrodynamic point of view does not see the underlying microscopic theory, we never explicitly consider particles in hydrodynamics and thus we cannot prove the existence of anomalies in an hydrodynamic setting, however we can still use the information gained from quantum field theory to account for anomalies in an hydrodynamic framework. In particular, we will claim that the hydrodynamic equations must be modified to account for both the external magnetic field and the new chiral anomaly in the form [49]

$$\partial_\mu T^{\mu\nu} = F^{\nu\lambda} J_\lambda \tag{4.1a}$$

$$\partial_\mu J^\mu = c E^\mu B_\mu \tag{4.1b}$$

where we defined $E^\mu = F^{\mu\nu} u_\nu$ and $B^\mu = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta} u_\nu F_{\alpha\beta}$[1] as the electric and magnetic field in the rest frame of the fluid [49], while c is a constant the describe the strength of the anomaly. This is exactly the result we reviewed and discussed in Chapter 2, i.e. a parallel electric and magnetic field induce an anomalous axial current due to the fact that new chiral fermions are produced in the system.

Not only the conservation laws of hydrodynamics, but also the constitutive equation for the current gains extra terms in the presence of a chiral anomaly. In normal hydrodynamics the constitutive relation for a vector current cannot contain terms made from the Levi-Civita tensor, since these terms explicitly break the parity invariance of the theory, but if the current is an axial current, then we can in principle add two first-order non-dissipative terms to the constitutive relation. Following the same procedure of the previous chapter, if the current is expressed as $J^\mu = \rho u^\mu + \nu^\mu$ then the most generic expression for first-order term ν^μ in the Landau frame is [49, 50]

$$\nu^\mu = -\sigma_E T \Delta^{\mu\nu} \partial_\nu \left(\frac{\mu}{T}\right) + \sigma_E E^\mu + \xi \omega^\mu + \xi^{(B)} B^\mu \tag{4.2}$$

where we have defined the vorticity $\omega^\mu = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta} u_\nu \partial_\alpha u_\beta$; $\xi^{(B)}$ and ξ are respectively the chiral magnetic and chiral vortical conductivity. These terms, while allowed by symmetries (the fact that it is an axial current) are still forbidden by the entropy production if no anomaly is present (the fact that the divergence of the entropy current has to be non-negative).

Following the usual procedure by Landau and Son [33, 49]: the equations of hydrodynamics are given in (4.1), from these, using the Landau matching

[1] The Levi-Civita tensor is defined with $\varepsilon_{0123} = -1$.

conditions $\tau^{\mu\nu}u_\nu = 0$, $\nu^\mu u_\mu = 0$ and Euler equation $\epsilon + p = sT + \mu\rho$ we obtain the usual expression for the entropy production, but this time there is an extra term due to the anomaly

$$\partial_\mu S^\mu = \partial_\mu\Big(su^\mu - \frac{\mu}{T}\nu^\mu\Big) = -\frac{\tau^{\mu\nu}}{T}\partial_\mu u_\nu - \nu^\mu\Big(\partial_\mu\frac{\mu}{T} - \frac{E_\mu}{T}\Big) - c\frac{\mu}{T}E^\mu B_\mu$$

If no anomaly is present ($c = 0$) the positivity of entropy production $\partial_\mu S^\mu \geq 0$ gives the usual expressions for the constitutive relations in the Landau frame, but if $c \neq 0$ then the last term can be positive or negative, overwhelming the other terms.

The most generic correction we can make to the constitutive relation is exactly the one given in (4.2). The expression of the two new conductivities ξ and $\xi^{(B)}$ is completely determined by the anomaly itself (eventually even the gravitational anomaly) and by the positivity of entropy production, leading to [45, 49, 50]

$$\xi = c\mu^2 + c_g T^2 - \frac{2\rho}{\epsilon + p}\left(\frac{c\mu^3}{3} + c_g\mu T^2\right) \tag{4.3a}$$

$$\xi^{(B)} = c\mu - \frac{1}{2}\frac{\rho}{\epsilon + p}(c\mu^2 + c_g T^2) \tag{4.3b}$$

where c_g is the parameter related to the gravitational anomaly. In what follows we will only work with a chiral anomaly, setting to zero the gravitational one $c_g = 0$.

If multiple conserved currents are present then the equations of hydrodynamics are

$$\partial_\mu T^{\mu\nu} = F_a^{\nu\lambda} J_{a\lambda} \tag{4.4a}$$

$$\partial_\mu J_a^\mu = c_{abc} E_b^\mu B_{c\mu} \tag{4.4b}$$

with c_{abc} the anomaly coefficient, that is totally symmetric under permutation of indices and $F_a^{\mu\nu}$ the gauge fields that couple to their respective current charge. The constitutive relations for the currents can be generalized and become (as usually in the Landau frame) [50]

$$J_a^\mu = \rho_a u^\mu + \sigma_{ab}\Big(E_b^\mu - T\Delta^{\mu\nu}\partial_\nu\frac{\mu_b}{T}\Big) + \xi_a\omega^\mu + \xi_{ab}^{(B)}B^{b\mu} \tag{4.5}$$

with ρ_a and μ_a the charge density and chemical potential of the various charges, while σ_{ab} is the symmetric conductivity matrix that, by means of positivity of entropy production, implies $\sigma_{(ab)}$ to be positive semi-definite.

The anomalous coefficients are given by

$$\xi_a = c_{abc}\mu_b\mu_c - \frac{2}{3}\rho_a c_{bcd}\frac{\mu_b\mu_c\mu_d}{\epsilon+p}$$
$$\xi^{(B)}_{ab} = c_{abc}\mu_c - \frac{1}{2}\rho_a c_{bcd}\frac{\mu_c\mu_d}{\epsilon+p}$$

4.2 The model

We now have all the tools to start the hydrodynamic computation. In the model we consider a Weyl semimetal with two Weyl nodes of opposite chirality, hence a left-handed current with chemical potential μ_L and a right-handed current with μ_R. Instead of working with left and right-handed currents, however, we move to a description with a vector and an anomalous axial current, defining $J^\mu = J^\mu_R + J^\mu_L$ and $J^\mu_5 = J^\mu_R - J^\mu_L$, i.e. the charges are $U(1)_V \times U(1)_A$. Furthermore we will apply a constant small magnetic field to the system along the z direction, so $F_{12} = -F_{21} = B$ and all the other entries of the electromagnetic field tensor are zero. We are interested in computing the thermoelectric matrix in linear response theory in the hydrodynamic limit $\mathbf{k} \to 0$ when also energy, momentum and charge conservation equations are explicitly broken by a small parameter $1/\tau$. The idea behind this method comes from [16].

As a remainder, we work in $1+3$ dimensions, in flat space, with metric $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$ and no gravitational anomaly. We will mostly follow [3] (section 4), but we will expand that work by calculating the full thermoelectric matrix.

4.2.1 The setup

We want to describe a system with two currents, but the hydrodynamic anomalous equations we showed in the previous section only work for one or three currents. The procedure to reduce the system from three currents to only two comes from [22]. The non anomalous terms are not a problem and can be found from (4.5) by picking σ_{ab} to be a 2×2 symmetric matrix. For the anomalous ξ_a and $\xi^{(B)}_{ab}$ terms instead, given three currents J^μ_a with $a = 1,2,3$ (and respective gauge fields $F^{\mu\nu}_a$) from equation (4.5), we can perform the following identifications to reduce to only two currents

$$\begin{array}{cc} A^A_\mu = A_{1\mu} & A^V_\mu = A_{2\mu} = A_{2\mu} \\ \mu_5 = \mu_1 & \mu = \mu_2 = \mu_3 \\ J^\mu_5 = J^\mu_1 & J^\mu = J^\mu_2 + J^\mu_3 \end{array}$$

and $c_{123} = c_{(123)} = c/2$. If the axial gauge field is set to zero $A^A_\mu = 0$ then the equations of hydrodynamics (4.4) simply become

$$\partial_\mu T^{\mu\nu} = F^{\nu\lambda} J_\lambda \tag{4.6a}$$
$$\partial_\mu J^\mu = 0 \tag{4.6b}$$
$$\partial_\mu J^\mu_5 = cE^\mu B_\mu \tag{4.6c}$$

By defining

$$\sigma^{(V)} = \xi_2 + \xi_3 \qquad \sigma^{(B)} = \xi^{(B)}_{22} + \xi^{(B)}_{33} + \xi^{(B)}_{23} + \xi^{(B)}_{32}$$
$$\sigma^{(V)}_5 = \xi_1 \qquad \sigma^{(B)}_5 = \xi^{(B)}_{12} + \xi^{(B)}_{13}$$

we can finally write the constitutive relations in the Landau frame for our two fluids model

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + p\Delta^{\mu\nu} + \tau^{\mu\nu} \tag{4.7a}$$
$$J^\mu = \rho u^\mu + \nu^\mu \tag{4.7b}$$
$$J^\mu_5 = \rho_5 u^\mu + \nu^\mu_5 \tag{4.7c}$$

where the dissipation terms are [50][2]

$$\tau^{\mu\nu} = -\eta\Delta^{\mu\alpha}\Delta^{\nu\beta}(\partial_\alpha u_\beta + \partial_\beta u_\alpha) - \Big(\zeta - \frac{2}{3}\eta\Big)\Delta^{\mu\nu}\partial_\alpha u^\alpha$$
$$\nu^\mu = -\sigma T\Delta^{\mu\nu}\partial_\nu\Big(\frac{\mu}{T}\Big) - \sigma_5 T\Delta^{\mu\nu}\partial_\nu\Big(\frac{\mu_5}{T}\Big) + \sigma E^\mu + \sigma^{(V)}\omega^\mu + \sigma^{(B)}B^\mu$$
$$\nu^\mu_5 = -\sigma_5 T\Delta^{\mu\nu}\partial_\nu\Big(\frac{\mu}{T}\Big) - \tilde{\sigma} T\Delta^{\mu\nu}\partial_\nu\Big(\frac{\mu_5}{T}\Big) + \sigma_5 E^\mu + \sigma^{(V)}_5\omega^\mu + \sigma^{(B)}_5 B^\mu$$

E^μ, B^μ and ω^μ are the electric field, magnetic field and vorticity defined earlier. The chemical potentials are defined as $\mu = \frac{1}{2}(\mu_R + \mu_L)$ while $\mu_5 = \frac{1}{2}(\mu_R - \mu_L)$ and similarly $\rho = \rho_R + \rho_L$ and $\rho_5 = \rho_R - \rho_L$. From the above identifications the anomalous conductivities are given by

$$\sigma^{(B)} = c\mu_5\Big(1 - \frac{\mu\rho}{\epsilon + p}\Big) \qquad \sigma^{(V)} = 2c\mu\mu_5\Big(1 - \frac{\mu\rho}{\epsilon + p}\Big)$$
$$\sigma^{(B)}_5 = c\mu\Big(1 - \frac{\mu_5\rho_5}{\epsilon + p}\Big) \qquad \sigma^{(V)}_5 = c\mu^2\Big(1 - \frac{2\mu_5\rho_5}{\epsilon + p}\Big)$$

It is interesting to notice that the first terms of $\sigma^{(B)}$ and $\sigma^{(B)}_5$ (respectively $c\mu_5$ and $c\mu$) are responsible for the chiral magnetic effect (and its symmetric counterpart on the axial current) (2.8). There are also the following relations

$$\epsilon + p = Ts + \mu\rho + \mu_5\rho_5 \qquad \mathrm{d}p = s\,\mathrm{d}T + \rho\,\mathrm{d}\mu + \rho_5\,\mathrm{d}\mu_5 \tag{4.9}$$

[2] In [3] they also set $\sigma = \tilde{\sigma}$, but we do not see any physical reason to do so, so we keep the expression more general.

from standard thermodynamics.

As our last step, we have to define the heat current generalizing the result from (3.29), that we will need to compute the thermoelectric matrix. In the case of multiple charges, when the system is perturbed with a small temperature change δT, the Hamiltonian changes by

$$\delta H = -\int \mathrm{d}^d x \left(\frac{\delta T(t,\mathbf{x})}{T} (\epsilon(t,\mathbf{x}) - \sum_{k=0}^{n} \mu_k \rho_k(t,\mathbf{x})) \right)$$

From this we are led to define the heat current as

$$\begin{aligned} Q^i = T^{0i} - \sum_{k=0}^{n} \mu_k J_k^i = T^{0i} - \mu_R J_R^i - \mu_L J_L^i = \\ = T^{0i} - \mu J^i - \mu_5 J_5^i \end{aligned} \tag{4.10}$$

where again the chemical potentials are meant to be taken at equilibrium.

4.2.2 The linearized equations

Now that we defined the problem we can proceed in the linear response theory framework. To do so, we first have to find the equilibrium solution and from there the equations that describe small (linear) fluctuations. The static equilibrium state solution is defined by the conditions $\mu(t,\mathbf{x}) = \mu$, $\mu_5(t,\mathbf{x}) = \mu_5$, $T(t,\mathbf{x}) = T$ and $u^\mu(t,\mathbf{x}) = (1,\mathbf{0})$. The equilibrium solution gives

$$T^{\mu\nu} = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \qquad \begin{aligned} J^\mu &= (\rho, 0, 0, B\sigma^{(B)}) \\ J_5^\mu &= (\rho_5, 0, 0, B\sigma_5^{(B)}) \end{aligned} \tag{4.11}$$

We now perturb the system by adding small fluctuations to the hydrodynamic variables

$$\mu(t,\mathbf{x}) = \mu + \delta\mu(t,\mathbf{x}) \tag{4.12a}$$

$$\mu_5(t,\mathbf{x}) = \mu_5 + \delta\mu_5(t,\mathbf{x}) \tag{4.12b}$$

$$T(t,\mathbf{x}) = T + \delta T(t,\mathbf{x}) \tag{4.12c}$$

$$u^\mu(t,\mathbf{x}) = (1, \delta u^i(t,\mathbf{x})) \tag{4.12d}$$

Furthermore, we also perturb the background with a small constant electric field $\delta F^{0i} = -\delta F^{i0} = \delta E^i$ in the $U(1)_V$ sector.

Applying this perturbation to the constitutive relations produce, to linear order in the fluctuations, the following elements

$$
\begin{aligned}
\delta T^{00} &= \delta\epsilon \\
\delta T^{0i} &= (\epsilon + p)\delta u^i \\
\delta T^{ij} &= \delta p\, \eta^{ij} - \eta\big(\partial^i \delta u^j + \partial^j \delta u^i - \frac{2}{3}\eta^{ij}\partial_k \delta u^k\big) - \zeta \eta^{ij}\partial_k \delta u^k \\
\delta J^t &= \delta\rho + \sigma^{(B)} B \delta u_z \\
\delta J^x &= \rho\delta u_x + \sigma(\delta F^{0x} + B\delta u_y) - \sigma T\partial_x\Big(\delta\frac{\mu}{T}\Big) - \sigma_5 T\partial_x\Big(\delta\frac{\mu_5}{T}\Big) + \\
&\qquad + \frac{1}{2}\sigma^{(V)}(\partial_y \delta u_z - \partial_z \delta u_y) \\
\delta J^y &= \rho\delta u_y + \sigma(\delta F^{0y} + B\delta u_x) - \sigma T\partial_y\Big(\delta\frac{\mu}{T}\Big) - \sigma_5 T\partial_y\Big(\delta\frac{\mu_5}{T}\Big) + \\
&\qquad - \frac{1}{2}\sigma^{(V)}(\partial_x \delta u_z - \partial_z \delta u_x) \\
\delta J^z &= \rho\delta u_z + \sigma\delta F^{0z} - \sigma T\partial_z\Big(\delta\frac{\mu}{T}\Big) - \sigma_5 T\partial_z\Big(\delta\frac{\mu_5}{T}\Big) + \\
&\qquad + \frac{1}{2}\sigma^{(V)}(\partial_x \delta u_y - \partial_y \delta u_x) + \delta\sigma^{(B)} B \\
\delta J_5^t &= \delta\rho_5 + \sigma_5^{(B)} B \delta u_z \\
\delta J_5^x &= \rho_5\delta u_x + \sigma_5(\delta F^{0x} + B\delta u_y) - \tilde{\sigma} T\partial_x\Big(\delta\frac{\mu_5}{T}\Big) - \sigma_5 T\partial_x\Big(\delta\frac{\mu}{T}\Big) + \\
&\qquad + \frac{1}{2}\sigma_5^{(V)}(\partial_y \delta u_z - \partial_z \delta u_y) \\
\delta J_5^y &= \rho_5\delta u_y + \sigma_5(\delta F^{0y} + B\delta u_x) - \tilde{\sigma} T\partial_y\Big(\delta\frac{\mu_5}{T}\Big) - \sigma_5 T\partial_y\Big(\delta\frac{\mu}{T}\Big) + \\
&\qquad - \frac{1}{2}\sigma_5^{(V)}(\partial_x \delta u_z - \partial_z \delta u_x) \\
\delta J_5^z &= \rho_5\delta u_z + \sigma_5\delta F^{0z} - \tilde{\sigma} T\partial_z\Big(\delta\frac{\mu_5}{T}\Big) - \sigma_5 T\partial_z\Big(\delta\frac{\mu}{T}\Big) + \\
&\qquad + \frac{1}{2}\sigma_5^{(V)}(\partial_x \delta u_y - \partial_y \delta u_x) + \delta\sigma_5^{(B)} B
\end{aligned}
$$

At the same time, the conservation equations to first order are similar to the ones in (4.7), but we add extra terms to account for the possible dissipative effects (these are also needed to make the DC longitudinal conductivity finite). Similar to the Drude model, these terms exists only outside of equilibrium

and imply a dissipation of energy, momentum and axial charge [3, 46]

$$\partial_\mu \delta T^{\mu 0} = \delta F^{0\mu} J_\mu + \frac{1}{\tau_e} \delta T^{\mu 0} u_\mu \tag{4.13a}$$

$$\partial_\mu \delta T^{\mu i} = \rho \delta F^{i0} + F^{i\mu} \delta J_\mu + \frac{1}{\tau_m} \delta T^{\mu i} u_\mu \tag{4.13b}$$

$$\partial_\mu \delta J^{\mu} = 0 \tag{4.13c}$$

$$\partial_\mu \delta J_5^{\mu} = c \delta E^\mu B_\mu + \frac{1}{\tau_c} \delta J_5^{\mu} u_\mu \tag{4.13d}$$

where all the terms without δ are meant to be taken at equilibrium, so that the expressions are consistently of leading order in the perturbation. τ_e is responsible for dissipation of energy, τ_m for dissipation of momentum and τ_c for dissipation of axial charge (i.e. the total electric charge is exactly conserved, and this is seen in the equation for J^μ, but the difference of right and left-handed fermion numbers has a dissipation term). Notice that τ_m could, in principle, be different depending on the direction, but here we consider the system to be isotropic as far as momentum dissipation goes.

Finally we substitute the expressions at linear order in the conservation equations above and we obtain the following set of equations

$$\big(\partial_t + \frac{1}{\tau_e}\big)\delta\epsilon + \partial_i[(\epsilon+p)\delta u^i] - \sigma^{(B)} B \delta F^{0z} = 0$$

$$\big(\partial_t + \frac{1}{\tau_m}\big)[(\epsilon+p)\delta u_x] + \partial_x \delta p - \eta(\partial_j^2 \delta u_x + \frac{1}{3}\partial_x \partial_j \delta u^j) - \zeta \partial_x \partial_j \delta u^j = \\ = \rho \delta F^{0x} + B \delta J_y$$

$$\big(\partial_t + \frac{1}{\tau_m}\big)[(\epsilon+p)\delta u_y] + \partial_y \delta p - \eta(\partial_j^2 \delta u_y + \frac{1}{3}\partial_y \partial_j \delta u^j) - \zeta \partial_y \partial_j \delta u^j = \\ = \rho \delta F^{0y} - B \delta J_x$$

$$\big(\partial_t + \frac{1}{\tau_m}\big)[(\epsilon+p)\delta u_z] + \partial_z \delta p - \eta(\partial_j^2 \delta u_z + \frac{1}{3}\partial_z \partial_j \delta u^j) - \zeta \partial_z \partial_j \delta u^j = \rho \delta F^{0z}$$

$$\partial_t(\delta\rho + \sigma^{(B)} B \delta u_z) + \partial_i(\rho \delta u^i + \sigma \delta F^{0i}) - \sigma T \partial_i^2\big(\delta \frac{\mu}{T}\big) + \\ - \sigma_5 T \partial_i^2\big(\delta \frac{\mu_5}{T}\big) + \sigma B(\partial_x \delta u_y - \partial_y \delta u_x) + \partial_z \delta \sigma^{(B)} B = 0$$

$$\big(\partial_t + \frac{1}{\tau_c}\big)(\delta\rho_5 + \sigma_5^{(B)} B \delta u_z) + \partial_i(\rho_5 \delta u^i + \sigma_5 \delta F^{0i}) - \tilde{\sigma} T \partial_i^2\big(\delta \frac{\mu_5}{T}\big) + \\ - \sigma_5 T \partial_i^2\big(\delta \frac{\mu}{T}\big) + \sigma_5 B(\partial_x \delta u_y - \partial_y \delta u_x) + \partial_z \delta \sigma_5^{(B)} B - cB\delta F^{0z} = 0$$

4.2.3 Laplace and Fourier transform

Before moving on we switch to a description that is explicitly in terms of the hydrodynamic variables only. This means that $\delta\epsilon$, δp, $\delta\rho$ and $\delta\rho_5$ are

expressed in terms of δT, $\delta\mu$ and $\delta\mu_5$. We have the following expressions (using Landsteiner's notations)

$$\begin{aligned}
\delta\epsilon &= \Big(\frac{\partial\epsilon}{\partial\mu_5}\Big)\Big|_{T,\mu}\delta\mu_5 + \Big(\frac{\partial\epsilon}{\partial\mu}\Big)\Big|_{T,\mu_5}\delta\mu + \Big(\frac{\partial\epsilon}{\partial T}\Big)\Big|_{\mu_5,\mu}\delta T = e_5\delta\mu_5 + e_1\delta\mu + e_2\delta T \\
\delta\rho &= \Big(\frac{\partial\rho}{\partial\mu_5}\Big)\Big|_{T,\mu}\delta\mu_5 + \Big(\frac{\partial\rho}{\partial\mu}\Big)\Big|_{T,\mu_5}\delta\mu + \Big(\frac{\partial\rho}{\partial T}\Big)\Big|_{\mu_5,\mu}\delta T = f_5\delta\mu_5 + f_1\delta\mu + f_2\delta T \\
\delta\rho_5 &= \Big(\frac{\partial\rho_5}{\partial\mu_5}\Big)\Big|_{T,\mu}\delta\mu_5 + \Big(\frac{\partial\rho_5}{\partial\mu}\Big)\Big|_{T,\mu_5}\delta\mu + \Big(\frac{\partial\rho_5}{\partial T}\Big)\Big|_{\mu_5,\mu}\delta T = s_5\delta\mu_5 + s_1\delta\mu + s_2\delta T \\
\delta p &= \rho_5\delta\mu_5 + \rho\delta\mu + s\delta T
\end{aligned}$$

Where all the e_i, f_i and s_i are thermodynamic parameters at equilibrium that depend on the system.

After substituting the above definitions we perform a complex Laplace transform on the equations of linearized hydrodynamics[3] in the time direction and a Fourier transform in the spatial direction, but we don't take the limit $\mathbf{k}\to 0$ yet. We are always free to create an electromagnetic field that has the form $\delta F^{0i}(t,\mathbf{x}) = \delta F^{0i}e^{-i\omega t + ik_j x^j}$ so that the external electric field is constant in the ω, $\mathbf{k}$ space. We also define

$$\omega_e = \omega + \frac{i}{\tau_e} \qquad \omega_m = \omega + \frac{i}{\tau_m} \qquad \omega_c = \omega + \frac{i}{\tau_c}$$

The time component of the stress energy tensor gives

$$\omega_e(e_1\delta\mu + e_2\delta T + e_5\delta\mu_5) - ie_2\delta T^{(0)} - iB\delta F^{0z}\sigma^{(B)} - (\epsilon+p)k_j\delta u^j + \cdots = 0$$

where the dots denote terms that are irrelevant to our calculation (they disappear after we perform the limit $\mathbf{k}\to 0$ or depend on perturbations we are not interested in, like $\delta\mu^{(0)}$), while the spatial components produce

$$\begin{aligned}
&(\epsilon+p)\omega_m\delta u_x - k_x(s\delta T + \rho\delta\mu + \rho_5\delta\mu_5) - i\rho\delta F^{0x} - iB\delta\tilde{J}_y + \cdots = 0 \\
&(\epsilon+p)\omega_m\delta u_y - k_y(s\delta T + \rho\delta\mu + \rho_5\delta\mu_5) - i\rho\delta F^{0y} + iB\delta\tilde{J}_x + \cdots = 0 \\
&(\epsilon+p)\omega_m\delta u_z - k_z(s\delta T + \rho\delta\mu + \rho_5\delta\mu_5) - i\rho\delta F^{0z} + \cdots = 0
\end{aligned}$$

where $\delta\tilde{J}_i$ stands for the Laplace-Fourier transform of δJ_i. The equations for the currents are

$$\begin{aligned}
\omega(f_1\delta\mu + f_2\delta T + f_5\delta\mu_5 + \sigma^{(B)}B\delta u_z) - if_2\delta T^{(0)} - k_i(\rho\delta u^i + \sigma\delta F^{0i}) + \\
- \sigma B(k_x\delta u_y - k_y\delta u_x) - k_z\delta\tilde{\sigma}^{(B)}B + \cdots = 0
\end{aligned}$$

[3] We defined $\mathcal{L}[f] = \int_0^\infty e^{-i\omega t}f(t)\,\mathrm{d}t$ that implies $\mathcal{L}[f'] = i\omega\mathcal{L}[f] + f(0^+)$.

and

$$\omega_c(s_1\delta\mu + s_2\delta T + s_5\delta\mu_5 + \sigma_5^{(B)}B\delta u_z) - is_2\delta T^{(0)} - k_i(\rho_5\delta u^i + \sigma_5\delta F^{0i}) + \\ + \sigma_5 B(k_x\delta u_y - k_y\delta u_x) - k_z\delta\tilde{\sigma}_5^{(B)}B - icB\delta F^{0z} + \cdots = 0$$

where again the $\delta\tilde{\sigma}^{(B)}$ means the Laplace-Fourier transform of $\delta\sigma^{(B)}$.

4.2.4 Solving the equations

As a final step, we solve the six equations above for the variables δT, $\delta\mu$, $\delta\mu_5$, δu_x, δu_y and δu_z in terms of the initial values of the hydrodynamic variables and external electromagnetic field. Since the equations are already very complicated, we set to zero all the initial data that are not of interest. Even with all these simplifications the solutions to the system of equations are extremely long and we cannot write them down explicitly in the general case.

We quote here the expression for the thermoelectric matrix (3.28) we defined in the previous chapter

$$\begin{pmatrix} J_i \\ Q_i \end{pmatrix} = \begin{pmatrix} \Sigma_{ij} & \alpha_{ij} \\ T\bar{\alpha}_{ij} & \bar{\kappa}_{ij} \end{pmatrix} \begin{pmatrix} E_j \\ -\nabla_j T \end{pmatrix} \tag{4.14}$$

We will just show how the general solution works by focusing on a specific example, the full explicit computation can be found in [51] as a *Mathematica* notebook. Say we want to work out the transport coefficient from $\delta Q_x = -\bar{\kappa}_{xy}\partial_y\delta T$. In Fourier space this equation becomes

$$\delta\tilde{Q}_x = -i\bar{\kappa}_{xy}k_y\delta T$$

After a Laplace-Fourier transform we find (3.30)

$$\delta\tilde{Q}_x(\omega) = -\frac{\bar{\kappa}_{xy}(\omega)}{i\omega}ik_y\delta T^{(0)}$$

In this case we can set to zero δF^{0i}, $\delta\mu^{(0)}$, $\delta\mu_5^{(0)}$ and $\delta u_i^{(0)}$ (we are not interested in the response of the system to these perturbations); k_x and k_z (since we only care about the term linear in k_y and then we take the limit $\mathbf{k} \to 0$ anyway); η and ζ (that are always coupled to second degree terms in spatial derivatives and thus go to zero in the small-$\mathbf{k}$ limit). Then we solve the six equations in term of the the only remaining initial data $\delta T^{(0)}$: the expressions for δT, $\delta\mu$, $\delta\mu_5$, δu_x, δu_y and δu_z will always be linear in $\delta T^{(0)}$, with the coefficients that are long functions of the magnetic field, the

thermodynamic variables at equilibrium (ρ, μ, T, ϵ, ...), the thermodynamic parameters (s_1, s_2, s_5, ...), k_y, the frequency ω and the dissipation time constants τ_e, τ_m and τ_c.

$$\begin{aligned}\delta\mu &= f_\mu(B,\omega,k_y)\delta T^{(0)} + \dots \\ \delta\mu_5 &= f_{\mu_5}(B,\omega,k_y)\delta T^{(0)} + \dots \\ \delta T &= f_T(B,\omega,k_y)\delta T^{(0)} + \dots \\ \delta u_i &= f_{u_i}(B,\omega,k_y)\delta T^{(0)} + \dots\end{aligned}$$

Where the f are rational functions with respect to k_y and the dots represent all the terms we can forget about and that we set to zero. Once we have the solution, we take the Laplace-Fourier transform of the heat current δQ_x (in the hydrodynamic variables) and we apply to it the conditions we listed above. This gives us

$$\delta\tilde{Q}_x = sT\delta u_x - B\delta u_y(\mu\sigma + \mu_5\sigma_5) - \frac{1}{2}ik_y\delta u_z(\mu\sigma^{(V)} + \mu_5\sigma_5^{(V)})$$

All is left to do now is to insert the solution we found in this expression of the heat current. This will automatically give us a term that is proportional to $\delta T^{(0)}$; we can have k_y approach zero, while keeping the leading order in k_y. The constant term will always be zero (as it should be, since no transport is present without a temperature gradient), so the leading order is the first-order term that gives us the transport coefficient we need. Notice that, in solving the systems with the electric field turned off, δu_z is always zero without momentum along k_z, while δu_x and δu_y are zero when computing the zz components of $\bar{\kappa}_{zz}$ and α_{zz}.

We can at least give the result to the system for Σ and $\bar{\alpha}$, that are the ones that depend on the electric field. In this case we can send to zero k_i at the start and solve the system of equations. This give us the solution [3]

$$\begin{aligned}\delta u_x &= \frac{B\delta F^{0y}(\rho^2+B^2\sigma^2) - i(\epsilon+p)(\delta F^{0x}\rho + B\delta F^{0y}\sigma)\omega_m}{B^4\sigma^2 - \omega_m^2(\epsilon+p)^2 + B^2[\rho^2 - 2i(\epsilon+p)\sigma\omega_m]} + \dots \\ \delta u_y &= \frac{B\delta F^{0x}(\rho^2+B^2\sigma^2) - i(\epsilon+p)(\delta F^{0y}\rho + B\delta F^{0x}\sigma)\omega_m}{B^4\sigma^2 - \omega_m^2(\epsilon+p)^2 + B^2[\rho^2 - 2i(\epsilon+p)\sigma\omega_m]} + \dots \\ \delta u_z &= \frac{i\rho\delta F^{0z}}{(\epsilon+p)\omega_m} + \dots\end{aligned}$$

$$\delta T = -\frac{B}{D}\left[\frac{i\sigma^{(B)}(f_5 s_1 - f_1 s_5)}{\omega_e} + \frac{i\rho[\sigma^{(B)}(e_5 s_1 - e_1 s_5) - \sigma_5^{(B)}(e_5 f_1 - e_1 f_5)]}{\omega_m(\epsilon + p)} + \right.$$
$$\left. + \frac{ic(e_5 f_1 - e_1 f_5)}{\omega_c}\right]\delta F^{0z} + \dots$$
$$\delta \mu = \frac{B}{D}\left[\frac{i\sigma^{(B)}(f_5 s_2 - f_2 s_5)}{\omega_e} + \frac{i\rho[\sigma^{(B)}(e_5 s_2 - e_2 s_5) - \sigma_5^{(B)}(e_5 f_2 - e_2 f_5)]}{\omega_m(\epsilon + p)} + \right.$$
$$\left. + \frac{ic(e_5 f_2 - e_2 f_5)}{\omega_c}\right]\delta F^{0z} + \dots$$
$$\delta \mu_5 = \frac{B}{D}\left[\frac{i\sigma^{(B)}(f_2 s_1 - f_1 s_2)}{\omega_e} + \frac{i\rho[\sigma^{(B)}(e_2 s_1 - e_1 s_2) - \sigma_5^{(B)}(e_2 f_1 - e_1 f_2)]}{\omega_m(\epsilon + p)} + \right.$$
$$\left. + \frac{ic(e_2 f_1 - e_1 f_2)}{\omega_c}\right]\delta F^{0z} + \dots$$

By applying this procedure to the full thermoelectric matrix we were able to find the full matrices: Σ_{ij}, α_{ij}, $\bar{\alpha}_{ij}$ and $\bar{\kappa}_{ij}$.

4.3 Results

We give now the results we found for the various conductivities with this method.

4.3.1 The thermoelectric conductivity matrix

The Σ matrix

The transverse coefficients are the same found by [3, 46] and we list them here

$$\Sigma_{xx} = \sigma\frac{\omega_m(\omega_m + i\gamma + i\omega_{cy}^2/\gamma)}{(\omega_m + i\gamma)^2 - \omega_{cy}^2} \tag{4.15a}$$
$$\Sigma_{xy} = -\frac{\rho}{B}\frac{\gamma^2 + \omega_{cy}^2 - 2i\gamma\omega_m}{(\omega_m + i\gamma)^2 - \omega_{cy}^2} \tag{4.15b}$$

where we defined $\omega_{cy} = \frac{B\rho}{\epsilon+p}$ and $\gamma = \frac{\sigma B^2}{\epsilon+p}$. Note that ω_{cy} is a cyclotron frequency that create a resonance visible only in ultrapure crystals, when $1/\tau_m \ll \omega_{cy}$, while γ is a damping frequency that exists because of the collisions between particles and antiparticles which are executing cyclotron orbits in opposite directions. It is obvious from isotropy that $\Sigma_{yy} = \Sigma_{xx}$ and from Onsager relations we also find that $\Sigma_{xy} = -\Sigma_{yx}$.

All the transport coefficients that are odd under the symmetry $z \to -z$ vanish, so we have $\Sigma_{xz} = \Sigma_{zx} = \Sigma_{yz} = \Sigma_{zy} = 0$. All the symmetries that we listed here will hold for the other transport matrices as well, so from now on we just list the xx, xy and zz terms, that are the only independent ones.

The longitudinal component was already calculated in [3] and we rewrite it here

$$\Sigma_{zz} = \sigma + \frac{iB^2 c\sigma^{(B)}}{\omega_e D} K_e + \frac{i\rho}{\omega_m(\epsilon+p)}\left[\rho - \frac{B^2 c}{D} K_0\right] + \frac{iB^2 c^2}{\omega_c D} K_c \tag{4.16}$$

with D the determinant

$$D = \begin{vmatrix} e_5 & e_2 & e_1 \\ f_5 & f_2 & f_1 \\ s_5 & s_2 & s_1 \end{vmatrix} \tag{4.17a}$$

We also define some other expressions that will be useful to write the results. We start with

$$D_e = \begin{vmatrix} f_5 & f_2 & f_1 \\ s_5 & s_2 & s_1 \\ \rho_5 & s & \rho \end{vmatrix} \qquad D_m = \begin{vmatrix} e_5 & e_2 & e_1 \\ s_5 & s_2 & s_1 \\ \rho_5 & s & \rho \end{vmatrix} \qquad D_c = \begin{vmatrix} e_5 & e_2 & e_1 \\ f_5 & f_2 & f_1 \\ \rho_5 & s & \rho \end{vmatrix} \tag{4.17b}$$

and

$$\begin{aligned}
K_e &= f_2 s_1 - f_1 s_2 - \frac{\rho}{\epsilon+p}[\mu(f_2 s_1 - f_1 s_2) + \mu_5(f_5 s_2 - f_2 s_5)] + \\
&\qquad + \frac{\mu\mu_5\rho}{(\epsilon+p)^2}[D + D_e] \\
K_c &= e_2 f_1 - e_1 f_2 - \frac{\rho}{\epsilon+p}[\mu(e_2 f_1 - e_1 f_2) + \mu_5(e_5 f_2 - e_2 f_5)] + \frac{\mu\mu_5\rho}{(\epsilon+p)^2} D_c \\
K_m &= e_1 s_2 - e_2 s_1 - \frac{\rho}{\epsilon+p}[\mu(e_1 s_2 - e_2 s_1) + \mu_5(e_2 s_5 - e_5 s_2)] + \\
&\qquad - \frac{\mu\mu_5\rho}{(\epsilon+p)^2} D_m - \frac{\mu\mu_5 D}{\epsilon+p} \\
K_0 &= \sigma^{(B)} K_m + \sigma_5^{(B)} K_c
\end{aligned}$$

We can see that all the dissipation terms are needed to have a finite DC longitudinal conductivity, furthermore it grows with B^2 as expected. For a discussion about this result see [3].

We also define (we will use these later)

$$K_e^5 = f_5 s_2 - f_2 s_5 - \frac{\rho_5}{\epsilon+p}[\mu(f_2 s_1 - f_1 s_2) + \mu_5(f_5 s_2 - f_2 s_5)] + \\ + \frac{\mu\mu_5\rho_5}{(\epsilon+p)^2}[D + D_e]$$

$$K_c^5 = e_5 f_2 - e_2 f_5 - \frac{\rho_5}{\epsilon+p}[\mu(e_2 f_1 - e_1 f_2) + \mu_5(e_5 f_2 - e_2 f_5)] + \\ + \frac{\mu\mu_5\rho_5}{(\epsilon+p)^2} D_c - \frac{\mu\mu_5 D}{\epsilon+p}$$

$$K_m^5 = e_2 s_5 - e_5 s_2 - \frac{\rho_5}{\epsilon+p}[\mu(e_1 s_2 - e_2 s_1) + \mu_5(e_2 s_5 - e_5 s_2)] - \frac{\mu\mu_5\rho_5}{(\epsilon+p)^2} D_m$$

$$K_0^5 = \sigma^{(B)} K_m^5 + \sigma_5^{(B)} K_c^5$$

Note that these last K^5 terms are the ones that appear in the expression for the longitudinal conductivity of the axial current Σ_{zz}^5, defined as $\delta J_5^z = \Sigma_{zz}^5 \delta F^{0z}$, in the same position of the K that appear in Σ_{zz}. Even if not an observable quantity, the expression for the axial conductivity is [3]

$$\Sigma_{zz}^5 = \sigma_5 + \frac{iB^2 c\sigma^{(B)}}{\omega_e D} K_e^5 + \frac{i\rho}{\omega_m(\epsilon+p)}\left[\rho_5 - \frac{B^2 c}{D} K_0^5\right] + \frac{iB^2 c^2}{\omega_c D} K_c^5 \quad (4.18)$$

The $\bar{\alpha}$ matrix

The xx and xy terms are the same found by [46] (with the extra μ_5 terms that appear because of the new definition of the heat current).

$$\bar{\alpha}_{xx} = \frac{s}{B}\frac{i\omega_{cy}\omega_m(1 - \rho(\gamma\mu + \gamma_5\mu_5)/(\omega_{cy}^2 sT))(\gamma - i\omega_m)}{(\omega_m + i\gamma)^2 - \omega_{cy}^2} \quad (4.19a)$$

$$\bar{\alpha}_{xy} = -\frac{s}{B}\frac{\gamma^2 + \omega_{cy}^2 - i\omega_m\left[\gamma - \rho(\gamma\mu + \gamma_5\mu_5)/(sT)\right]}{(\omega_m + i\gamma)^2 - \omega_{cy}^2} \quad (4.19b)$$

with $\gamma_5 = \frac{B^2\sigma_5}{\epsilon+p}$ (from Hartnoll results, switching each $\gamma\mu$ with $\gamma\mu + \gamma_5\mu_5$ gives the two-currents expressions). The zz component instead has the same form of Σ_{zz}

$$\bar{\alpha}_{zz} = -\frac{\mu\sigma + \mu_5\sigma_5}{T} + \frac{iB^2 c\sigma^{(B)}}{\omega_e D} A_e + \frac{i\rho}{\omega_m(\epsilon+p)}\left[s - \frac{B^2 c}{D} A_0\right] + \frac{iB^2 c^2}{\omega_c D} A_c \quad (4.20)$$

where we also defined

$$A_e = -\frac{\mu K_e + \mu_5 K_e^5}{T}$$
$$A_0 = -\frac{\mu K_0 + \mu_5 K_0^5}{T}$$
$$A_c = -\frac{\mu K_c + \mu_5 K_c^5}{T}$$

Again, we see that all the dissipations are needed for the conductivity to be bounded in DC. From this expression we see that $\bar{\alpha}_{zz}$ is closely related to Σ_{zz} and Σ^5_{zz}, in fact we can write

$$\bar{\alpha}_{zz} = -\frac{\mu\Sigma_{zz} + \mu_5\Sigma^5_{zz}}{T} + \frac{i\rho}{T\omega_m}$$

This is nothing but the usual Ward identity found in literature, appropriately generalized for a fluid with two components [52].

The α matrix

For this matrix we find

$$\alpha_{xx} = \frac{s}{B}\frac{\omega_m}{D[(\omega_m + i\gamma)^2 - \omega_{cy}^2]}\left(W_0 + \frac{\omega W_c}{\omega_c} + \frac{\omega W_e}{\omega_e}\right) \tag{4.21a}$$

$$\alpha_{xy} = -\frac{s}{B}\frac{iB}{D[(\omega_m + i\gamma)^2 - \omega_{cy}^2]}\left(Z_0 + \frac{\omega Z_c}{\omega_c} + \frac{\omega Z_e}{\omega_e}\right) \tag{4.21b}$$

$$\alpha_{zz} = G_0 + cB^2 G_1 \tag{4.21c}$$

G_0 is the non-anomalous term (its expression is still rather long), while G_1 is a very long expression (many screens long in *Mathematica*) that depends on the anomaly. W_i and Z_i are long expressions containing ω, B, the thermodynamic variables and the time constants. All these results that could not be written here due to their length can be found in [51].

The $\bar{\kappa}$ matrix

And finally

$$\bar{\kappa}_{xx} = -\frac{1}{D[(\omega_m + i\gamma)^2 - \omega_{cy}^2]}\left(R_0 + \frac{\omega R_c}{\omega_c} + \frac{\omega R_e}{\omega_e}\right) \tag{4.22}$$

$$\bar{\kappa}_{xy} = -\frac{1}{D[(\omega_m + i\gamma)^2 - \omega_{cy}^2]}\left(N_0 + \frac{\omega N_c}{\omega_c} + \frac{\omega N_e}{\omega_e}\right) \tag{4.23}$$

$$\bar{\kappa}_{zz} = Y_0 + cB^2 Y_1 \tag{4.24}$$

where again the R_i, N_i and Y_i are long expressions that depend on all the variables and parameters of the problem [51].

4.3.2 Constraints from Onsager reciprocal relations

As can be seen from the form of the above expressions, the α and $\bar{\alpha}$ matrices are different. This is not possible if we also want our microscopic theory to obey Onsager relations: from the derivation we saw in the previous chapter α should be equal to $\bar{\alpha}$, when this is not the case, it means that there are certain constraints that must be imposed on the expressions such that they are equal.

Furthermore, the xx and xy components of α and $\bar{\kappa}$ should reduce to the usual magnetohydrodynamic expressions [46]: these terms do not see the anomaly and should behave normally. For the same reason, the non-anomalous term of the longitudinal conductivities should also reduce to the usual hydrodynamic expression (this non-anomalous term cannot see the magnetic field at all, so it reduces to the pure hydrodynamic form).

The usual procedure to fix these problems is to work in DC (the constraints must work for all possible values of the frequency, in particular $\omega = 0$ tends to simplify the expressions a bit) and to make some assumptions on the form on the parameters such that the two matrices become equal. We will now focus on the first component, the xx one.

As we mentioned, $\bar{\alpha}_{xx}$ has the usual magnetohydrodynamic expression [46], so we will look for a set of constraints that make $\alpha \to \bar{\alpha}$, instead of looking for constraints that make them equal, but different from the usual magnetohydrodynamic result. We also observe that in DC the two terms W_e and W_c of α_{xx} disappear because of the ω factor in front (4.21a), furthermore we note that the denominator of α_{xx} and $\bar{\alpha}_{xx}$ are identical, apart for the factor D, so we can just focus on the numerator while remembering to divide by D the terms from α.

We work under certain assumptions:

- We clearly do not want any constraints on the thermodynamic variables at equilibrium (ρ, s, μ, etc...) since this would change the expression for $\bar{\alpha}$, so we will mostly look for constraints on the equilibrium parameters e_i, f_i and s_i (there are in total 9 of these parameters, but we can expect physical systems at equilibrium to be described by fewer degrees of freedom).

- τ_m is a parameter specific to our model and it is not even relevant once the equilibrium state is reached (it only exists out of equilibrium to dampen the fluctuations), so it cannot enter any constraint we find; if

we consider the expressions for α_{xx} and $\bar{\alpha}_{xx}$ as a polynomial in τ_m, we can then ask that coefficients of equal degree must be the same.

- A similar way of thought allows us to separate terms with or without the magnetic field B (we asked that B has to be small, much smaller than the temperature and chemical potential, in fact it does not appear in the Euler relation as mB).

- The transport coefficients σ and σ_5 cannot enter the thermodynamic equilibrium equations, so we can again consider separately coefficients of σ and σ_5 from the two thermoelectric conductivities and match them one-to-one.

The first condition comes from requiring that the coefficients of τ_m but without B (or, equivalently, without conductivities) are equal. As we said, focusing on the numerator and simplifying the common factors we find

$$\frac{-f_2 D_m}{D} = s$$

where D_m and D are the determinant we defined earlier on in (4.17). The LHS comes from α, while the RHS from $\bar{\alpha}$. The second condition comes from the request that terms without τ_m must be equal, in particular asking that the coefficients of σ and σ_5 are the same. For the σ coefficient this gives us

$$\frac{f_2[s_5(e_2 T + e_1 \mu) - e_5(s_2 T + s_1 \mu)]}{D} = -\mu$$

For the σ_5 coefficient instead

$$\frac{f_2[s_1(e_2 T + e_5 \mu_5) - e_1(s_2 T + s_5 \mu_5)]}{D} = \mu_5$$

where again we kept the term that comes from α on the LHS and from $\bar{\alpha}$ on the RHS. The last condition one could hope to extract, the requirement that the coefficients of $\tau_m B^2$ must be equal, leads to these two last exact equations.

The only solution we were able to find that obeys the above conditions is an unphysical one

$$s_2 = 0 \qquad \text{and} \qquad e_2 = 0$$

While $s_2 = 0$ could be argued to make some sense (it requires that changing the equilibrium temperature does not change the difference between the density of right and left-handed fermions) $e_2 = 0$ is very hard to justify: e_2 is the specific heat and it implies that changing the temperature does not

increase the energy of the system. Furthermore, we know that the entropy is related to the specific heat by $s = \frac{\partial p}{\partial T} = \frac{\partial p}{\partial \epsilon}\frac{\partial \epsilon}{\partial T} = v_s^2 e_2$, with v_s the speed of sound, so $e_2 = 0$ implies $s = 0$ (this relation is relevant only when $e_2 = 0$, since we do not know v_s anyway).

4.3.3 Maxwell relations

The unphysical result we obtained is a strong signal that something is not right in our procedure. The conditions we imposed seem reasonable, so we look for other relations, i.e. Maxwell relations, that could potentially help us further reduce the number of degrees of freedom of the equilibrium thermodynamics.

We are working in the grand canonical ensemble, with T, μ and μ_5 as our thermodynamic independent variables. The thermodynamic potential suited for this setup is the Landau grand canonical potential $\Omega = -pV$, so the Landau potential (volume) density is just $-p$. We now differentiate p twice, with respect to the thermodynamic variables, and from Schwarz's theorem we ask that the second derivatives do not change upon changing the order of the derivative. Deriving p with respect to μ and μ_5 gives us the Maxwell relation

$$\left.\frac{\partial^2 p}{\partial \mu \partial \mu_5}\right|_T = \left.\frac{\partial^2 p}{\partial \mu_5 \partial \mu}\right|_T \qquad \Longrightarrow \qquad f_5 = s_1$$

Deriving p with respect to T and μ (and μ_5) gives us two further constraints, even if apparently less relevant

$$\left.\frac{\partial^2 p}{\partial T \partial \mu}\right|_{\mu_5} = \left.\frac{\partial^2 p}{\partial \mu \partial T}\right|_{\mu_5} \qquad \Longrightarrow \qquad \left.\frac{\partial \rho}{\partial T}\right|_{\mu_5,\mu} = \left.\frac{\partial s}{\partial \mu}\right|_{\mu_5,T} = f_2$$

$$\left.\frac{\partial^2 p}{\partial T \partial \mu_5}\right|_{\mu} = \left.\frac{\partial^2 p}{\partial \mu_5 \partial T}\right|_{\mu} \qquad \Longrightarrow \qquad \left.\frac{\partial \rho_5}{\partial T}\right|_{\mu_5,\mu} = \left.\frac{\partial s}{\partial \mu_5}\right|_{\mu,T} = s_2$$

We can also use the same derivatives of p in conjunction with Euler equation, to find two further relations between the thermodynamic parameters. We derive p with respect to μ using Euler equation $p = sT + \mu\rho + \mu_5\rho_5 - \epsilon$ and the Maxwell relations found above and we get

$$\begin{aligned}
\frac{\partial \cancel{p}}{\cancel{\partial}\mu} &= \frac{\partial s}{\partial \mu}T + \frac{\partial \rho}{\partial \mu}\mu + \cancel{\rho} + \frac{\partial \rho_5}{\partial \mu}\mu_5 - \frac{\partial \epsilon}{\partial \mu} \\
e_1 &= \mu f_1 + \mu_5 s_1 + \frac{\partial s}{\partial \mu}T \\
e_1 &= \mu f_1 + \mu_5 s_1 + T f_2
\end{aligned}$$

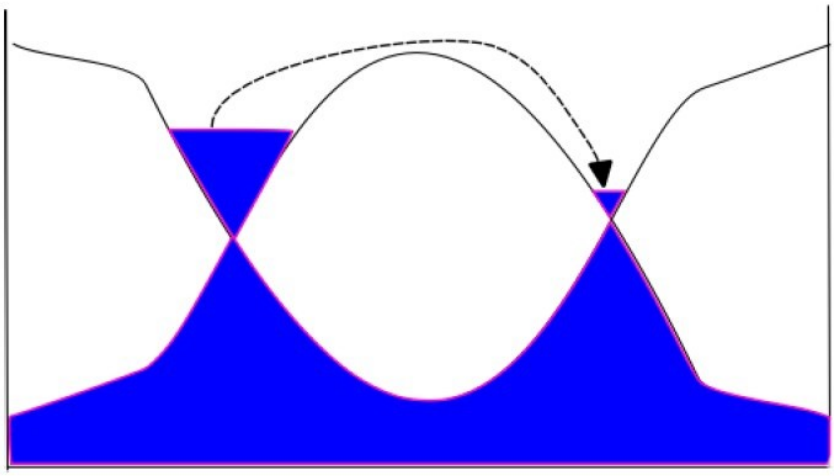

Figure 4.1: Image from [3]. A schematic representation of a scattering event: because of inter-valley scattering a particle from the left node moves to the right one. An event of this kind, only possible in the presence of a chiral anomaly, dissipate both axial charge and energy.

Applying the same procedure while deriving with respect to μ_5 gives instead

$$e_5 = \mu f_5 + \mu_5 s_5 + T s_2$$

These relations are constraints on the thermodynamic parameters that come from the thermodynamics itself, so these are independent on the assumptions we made above to fix the Onsager relations. These expressions however, when applied to α_{xx} and α_{xy}, keep producing the same conditions $e_2 = s_2 = 0$ and seem to not be helpful in fixing Onsager relations.

4.3.4 Fixing the transverse components

The procedure shown above does not lead to anything useful, yet we need to find a way to reduce α_{xx} and α_{xy} to the usual magnetohydrodynamic expressions (and also $\bar{\kappa}_{xx}$ and $\bar{\kappa}_{xy}$).

Our procedure reproduces exactly the results from Hartnoll [46] and Landsteiner [3] when we use their exact setup, so we are confident that the procedure is correct. Yet we notice that, even in the very simple case of a single vector current without anomaly, we do not find the usual magnetohydrodynamic conductivities for α and $\bar{\kappa}$ if we keep the dissipation of charge and energy turned on.

We argue that this is because the axial charge and energy dissipations are there only when the anomaly is present: without anomaly energy is conserved (we will explain later on how the anomaly can dissipate energy) and so is the fermion number for each chirality.

This means that τ_e and τ_c can only be present for scattering events with momentum along k_z: all the dynamics that happen along k_x and k_y cannot depend on the dissipation of energy and charge. Furthermore, we claim that, even when the momentum is in the direction k_z, not all the scattering events can dissipate energy and charge, but only those that depend on the anomaly. We can express this, without being too rigorous, by claiming that the time constants are actually functions that depend on the momentum along k_z and the anomaly constant c

$$\tau_{e/c} \sim k_z^{-m} c^{-n} \qquad \text{with} \qquad m, n > 0$$

So that it is now clear that when k_z is sent to zero for the xx and xy components or when we are computing the non-anomalous term of zz (so with $c \to 0$) then the time constants go to infinity and the terms with $1/\tau_{e/c}$ disappear from the linearized system of equations.

Another possible way to think about this is the to use a perturbative-like approach (it is not a true perturbative approach, in the sense that the correction could be bigger than the starting value). We could imagine to compute the thermoelectric matrix of a non-anomalous two-currents fluid by sending to zero c, τ_e and τ_c; this way we obtain the usual magnetohydrodynamic results found in literature. From these results we now turn on the anomaly, so $c \neq 0$ and we also add the relaxation constants for the energy and axial charge. We know that this new terms cannot have any effect on the non-anomalous coefficients, so in our calculation we can explicitly focus on the anomalous correction and then we add these to the results obtained at $c = 0$.

τ_c is the constant that describes the scattering events that are explicitly there because of the anomaly: when a particle scatters from one node to the other it changes chirality, thus dissipating chiral charge, see Figure 4.1. This process was already introduced in the chapter about Weyl semimetals when explaining the negative magnetoresistance: a steady axial current implies a balance between the chiral fermions production caused by the anomaly and the axial charge dissipations due to the scattering events.

To understand why a chiral system with anomaly also dissipate energy is a little trickier, but we will follow here the argument from [3]. Consider an inter-valley scattering like the one pictured in 4.1, i.e. with a fermion that

scatters from one node to the other of opposite chirality. In the presence of parallel electric and magnetic fields the local Fermi energies of the two Weyl cones will be shifted because of the injection of axial charge via the chiral anomaly $\frac{\mathrm{d}\rho_5}{\mathrm{d}t} = 1/(2\pi^2)\mathbf{E}\cdot\mathbf{B}$. The difference in the local Fermi energies of the two nodes is determined by the axial chemical potential μ_5. An inter-valley scattering event changes the axial density $\rho_5 = \rho_R - \rho_L$, but this implies an energy cost of the form $\delta\epsilon = \mu_5\delta\rho_5$. This schematic representation explains why we should consider energy relaxation in our model and how it is indeed important to have finite DC conductivities.

On a practical level, this means that we must sent to infinity τ_e and τ_c when computing the expressions for the xx and xy components and the same must be true for the non-anomalous part of the zz component.

This ad hoc correction seems to be relevant only for α and $\bar{\kappa}$, while Σ and $\bar{\alpha}$ reproduce the correct non-anomalous magnetohydrodynamic result even while keeping all the dissipations on, because the τ_e and τ_c disappear from the system of equations when the momentum $\mathbf{k}$ is set to zero, except for the anomalous term.

With this idea in mind we can finally compute all the thermoelectric conductivities and we finally obtain the usual magnetohydrodynamic results for all the transverse conductivities (except for the fact that we are working with two currents with different transport coefficients).

4.3.5 Final results

With this correction in mind, we can finally write down the expressions for the solution to the system of equations when we only keep k_x and k_y turned on. The solution is already given to leading order in k_x

$$\delta u_x = \frac{B^2(sT\sigma + \mu\rho\sigma + \mu_5\rho\sigma_5) - isT\omega_m(\epsilon+p)}{B^4\sigma^2 - \omega_m^2(\epsilon+p)^2 + B^2[\rho^2 - 2i(\epsilon+p)\sigma\omega_m]}\frac{k_x\delta T^{(0)}}{T\omega} + \dots$$
$$\delta u_y = \frac{-sT\rho + (\mu\sigma + \mu_5\sigma_5)[B^2\sigma - i(\epsilon+p)\omega_m]}{B^4\sigma^2 - \omega_m^2(\epsilon+p)^2 + B^2[\rho^2 - 2i(\epsilon+p)\sigma\omega_m]}\frac{Bk_x\delta T^{(0)}}{T\omega} + \dots$$
$$\delta T = \frac{i\delta T^{(0)}}{\omega} + \dots$$

where the dots are terms of higher powers of k_x and the terms not listed (δu_z, $\delta\mu$ and $\delta\mu_5$) are zero at linear order. The solution for the k_y case is similar and is obtained by switching $\delta u_x \to \delta u_y$ and $\delta u_y \to (-\delta u_x)$ (and also $k_x \to k_y$).

We can finally write here in a more explicit form the coefficients that we could not list earlier because of their length. α_{xx} and α_{xy} are now the same found in (4.19a) and (4.19b).

Furthermore we can also list here the $\bar{\kappa}_{xx}$ and $\bar{\kappa}_{xy}$ components. The form of this result is very different by the one found in [46], but this is because the second current mixes things up and it is not straightforward to generalize the usual expression found in literature; in the one current limit ($\mu_5 = \rho_5 = 0$) one obtains the usual result. $\bar{\kappa}_{xx}$ is

$$\bar{\kappa}_{xx} = \frac{[B^2\sigma - i\omega_m(\epsilon+p)][s^2T^2 - B^2(\mu\sigma+\mu_5\sigma_5)^2] + 2B^2 sT\rho(\mu\sigma+\mu_5\sigma_5)}{T[B^4\sigma^2 - \omega_m^2(\epsilon+p)^2 + B^2[\rho^2 - 2i(\epsilon+p)\sigma\omega_m]]} + \\ - i\frac{\mu^2\sigma + 2\mu\mu_5\sigma_5 + \mu_5^2\tilde{\sigma}}{T}$$

Even if not explicit in this form, the denominator of the first term is the usual denominator found for the other conductivities $(\omega_m + i\gamma)^2 - \omega_{cy}^2$ with an extra factor $-(\epsilon+p)$.

$$B^4\sigma^2 - \omega_m^2(\epsilon+p)^2 + B^2[\rho^2 - 2i(\epsilon+p)\sigma\omega_m] = -(\epsilon+p)[(\omega_m+i\gamma)^2 - \omega_{cy}^2]$$

The $\bar{\kappa}_{xy}$ component is

$$\bar{\kappa}_{xy} = B\frac{s^2T^2\rho - B^2\rho(\mu\sigma+\mu_5\sigma_5) - 2sT(\mu\sigma+\mu_5\sigma_5)[B^2\sigma - i\omega_m(\epsilon+p)]}{T[B^4\sigma^2 - \omega_m^2(\epsilon+p)^2 + B^2[\rho^2 - 2i(\epsilon+p)\sigma\omega_m]]}$$

The zz components for α and $\bar{\kappa}$ are given here

$$\alpha_{zz} = -\frac{\mu\sigma+\mu_5\sigma_5}{T} + \frac{is\rho}{\omega_m(\epsilon+p)} + cB^2G_1(\omega_e,\omega_c,\omega_m)$$
$$\bar{\kappa}_{zz} = \frac{\mu^2\sigma + 2\mu\mu_5\sigma_5 + \mu_5^2\tilde{\sigma}}{T} + \frac{is^2T}{\omega_m(\epsilon+p)} + cB^2Y_1(\omega_e,\omega_c,\omega_m)$$

where the anomalous terms are still the complicated expressions that can be found in [51], while the non-anomalous terms have the usual hydrodynamic form. This means that, even if we managed to fix the xx and xy components of α, the zz component is not Onsager reciprocal $\bar{\alpha}_{zz} \neq \alpha_{zz}$. In [52] the authors also found that the two expressions are not automatically Onsager reciprocal from a simpler but similar model.

It should be noted that the anomalous expressions for the α_{zz} and $\bar{\kappa}_{zz}$ still have some issues: they do not have the usual form found for the longitudinal electrical conductivity of a sum of three terms $T_e/\omega_e + T_c/\omega_c + T_m/\omega_m$, like Σ_{zz} and $\bar{\alpha}_{zz}$, instead they still have a term proportional to $1/\omega$ (remember that the linearized equation for the vector current is the only one that has

ω without dissipation constants), but they also present terms that have ω at the numerator and thus that disappear in the DC limit (similar to the expressions for α_{xx} and α_{xy} before the fix (4.21a), (4.21b)).

These peculiar forms arise for both $\bar{\kappa}_{zz}$ and α_{zz}, so it is likely that the problem lies in the solution to the system of equations itself, rather than in the specific form of the thermoelectric conductivities.

We want to emphasize the fact that these problems are all related to the presence of the dissipation of charge and energy. The procedure we followed in this work by Hartnoll [46] seems reasonable and it produces the expected results when only dissipation of momentum is present, however the same method leads to incorrect expressions for the conductivities when multiple dissipations are present. To check this statement we performed the same exact computation as above, but this time with $\tau_e = \tau_c \to \infty$. This test showed that indeed, without energy and charge dissipations, the longitudinal anomalous conductivities that couple to the temperature gradient α_{zz} and $\bar{\kappa}_{zz}$ seems to be well behaved, in the sense that the expressions simplify to something that resemble Σ_{zz} and $\bar{\alpha}_{zz}$ (albeit with ω instead of ω_e and ω_c).

As we said earlier α_{ij} and $\bar{\kappa}_{ij}$ are found by keeping the leading order in k_j after substituting the solution to the system of equations in the expressions for δJ_i and δQ_i. This is exact for xx and xy, even before the fix, but it is important to note that for the zz component there is also an anomalous constant term that is non vanishing even at $k_z = 0$. For example, the electric current has this term at $k_z = 0$

$$\delta J_z\big|_{k_z=0} = -\frac{iBc\mu\mu_5[f_2(\epsilon+p) - (e_2+s)\rho]}{(\epsilon+p)^2}\delta T^0$$

and a similar one for δQ_z.

These terms, that seem to indicate some transport without a temperature gradient, are strongly dependent on the form of the anomalous coefficients. To find the expressions for $\sigma^{(B)}$ and $\sigma_5^{(B)}$ (and also $\sigma^{(V)}$ and $\sigma_5^{(V)}$, but these are not relevant to our calculation and they always disappear from the expressions) we used the method after [22] to reduce from three to two currents. However, it is possible to find the same result, to first order in μ and μ_5, by working with two currents (one vector and one axial current) since the beginning and obtaining the anomalous coefficients following the same procedure Son [49] used for the single current case. This was done by [23] and they found that, to leading order in μ, μ_5

$$\sigma^{(B)} = c\mu_5 \qquad \text{and} \qquad \sigma_5^{(B)} = c\mu$$

in agreement with [22] results. Using these leading-order expressions for the anomalous conductivities simplifies the anomalous transport equations a lot (the main reason why the algebra becomes so complicated lies in the first-order expansion of the anomalous coefficients $\delta\sigma^{(B)}$) and, in particular, the above terms found at $k_z = 0$ disappear (thus α_{zz} and $\bar{\kappa}_{zz}$ are really the leading-order expressions in k_z).

4.4 Ward identities and special cases

We managed to find some accurate expressions for Σ_{zz} and $\bar{\alpha}_{zz}$, while the results found for α_{zz} and $\bar{\kappa}_{zz}$ still have some issues and in the present form they are way too long to be fully showed in here [51].

To recover the correct expressions for the conductivities the idea is to use Ward identities. Ward identities arise naturally from the equations of motion (in our case, from the conservation equations) when applied to the n-point functions of the theory. The one-point Ward identities are simply the conservation equations for the stress energy tensor and for the currents, while the two-point functions link together the correlation functions $G^R_{ij}(\omega, \mathbf{k})$.

In Chapter 3, for example, we used the Ward identities for the retarded correlation functions to shift our focus from $G^R_{F;\rho}$ and $G^R_{F;\epsilon-\mu\rho}$ to $G^R_{F;\mathbf{J}}$ and $G^R_{F;\mathbf{Q}}$, using the fact that $k^\mu G^R_{F;J^\mu} = 0$ [32, 46] (in the limit $\mathbf{k} \to 0$).

There are however other Ward identities that can be found by differentiating the one-point functions with respect to the gauge fields, this produces new relations between the two-point functions. In particular, since the two-point functions are closely related to the conductivities (3.31), these identities become a constraint that the conductivities must obey [53]. Usually these relations are used as a consistency check on the model, however here we can use them to actually find the correct expressions for $\bar{\kappa}_{zz}$.

The xx and xy components of α and $\bar{\kappa}$ (in this discussion we are already implying that Onsager is satisfied, so $\alpha = \bar{\alpha}$) are related to Σ by Ward identities that can be found in [53]; also the zz components obey a very simple Ward identity, specifically this is usually written as (we are still considering the magnetic field along z)

$$\alpha_{zz} = -\frac{\mu}{T}\Sigma_{zz} + \frac{i\rho}{\omega T}$$

In [52] Abbasi found the above relation to be true in his one-current model of a chiral fluid, if ω is switched with ω_m. In this thesis we already

showed (for $\bar{\alpha}_{zz}$, that is different from α_{zz} in this work)

$$\bar{\alpha}_{zz} = -\frac{\mu\Sigma_{zz} + \mu_5\Sigma^5_{zz}}{T} + \frac{i\rho}{\omega_m T} \tag{4.25}$$

that is in perfect agreement with [52] and [53]. The presence of ω_m instead of ω is an expected result already well documented, also the extra $\mu_5\Sigma^5_{zz}$ term generalizes naturally from our definition of the heat current.

Following the same exact argument, it is a well known result in literature the Ward identity that relates $\bar{\kappa}_{zz}$ with Σ_{zz} [53]

$$\bar{\kappa}_{zz} = \frac{\mu^2}{T}\Sigma_{zz} + \frac{i(\epsilon + p - 2\mu\rho)}{\omega T}$$

where again in [52] the authors found the same relation, but with ω_m instead of ω.

First of all, since $\bar{\alpha}_{zz}$ correctly obeys the Ward identity, from now on in this section we will consider $\bar{\alpha}_{zz}$ to be the correct expression for $\bar{\alpha}_{zz} = \alpha_{zz}$, so any argument or discussion that follows is based on the assumption that, after fixing the expressions and applying Onsager relation to the longitudinal conductivities, we end up with $\bar{\alpha}_{zz}$ as the thermoelectric conductivity.[4]

Secondly, we now claim that $\bar{\kappa}_{zz}$ must obey a relation that is a two-current generalization of the usual expression found in literature. To find this Ward identity, instead of deriving it from scratch, we can use the non-anomalous part of the conductivities to understand how they are linked and then assume that the same equation must be true even when we add the anomalous term (this is exactly what happens in (4.25), where the identity does not gain any extra term because of the anomaly; the anomaly content is all inside of Σ_{zz} and Σ^5_{zz}).

The non anomalous expressions (i.e. $B = 0$ or, equivalently, $c = 0$) for the conductivities are (we also list here Σ^5_{zz} since, as we saw for α_{zz}, it will appear in the Ward identity for $\bar{\kappa}_{zz}$)

$$\begin{aligned}
\Sigma_{zz} &= \sigma + \frac{i\rho^2}{\omega_m(\epsilon + p)} \\
\Sigma^5_{zz} &= \sigma_5 + \frac{i\rho\rho_5}{\omega_m(\epsilon + p)} \\
\bar{\alpha}_{zz} = \alpha_{zz} &= -\frac{\mu\sigma + \mu_5\sigma_5}{T} + \frac{i\rho s}{\omega_m(\epsilon + p)} \\
\bar{\kappa}_{zz} &= \frac{\mu^2\sigma + 2\mu\mu_5\sigma_5 + \mu_5^2\tilde{\sigma}}{T} + \frac{is^2T}{\omega_m(\epsilon + p)}
\end{aligned}$$

[4]Note that, even if Onsager relations might require $\tau_e = \tau_c = \tau_m$, like in [52], it is still worth keeping them with different names to better understand the role of each term.

We immediately notice that $\bar{\kappa}_{zz}$ contains a term with $\tilde{\sigma}$, but $\tilde{\sigma}$ does not appear in any other longitudinal conductivity. This means that $\tilde{\sigma}$ is not a free parameter, instead it must be related to the other conductivities. Following [3] we then claim that, guided by the symmetry of the expression, $\tilde{\sigma} = \sigma$.

With this definition we can finally write the Ward identity for the non-anomalous part and that is

$$\bar{\kappa}_{zz} = \frac{(\mu^2 + \mu_5^2)\Sigma_{zz} + 2\mu\mu_5\Sigma_{zz}^5}{T} + \frac{i[s^2T - \rho^2(\mu^2 + \mu_5^2) - 2\mu\mu_5\rho\rho_5]}{\omega_m T(\epsilon + p)} \quad (B = 0)$$

We now claim that this result holds in the most general case $B \neq 0$. Then the corrected expression for $\bar{\kappa}_{zz}$ is

$$\bar{\kappa}_{zz} = \frac{(\mu^2 + \mu_5^2)\sigma + 2\mu\mu_5\sigma_5}{T} + \frac{iB^2c\sigma^{(B)}}{\omega_e D}Y_e + \\ + \frac{i}{\omega_m(\epsilon + p)}\left[s^2T - \frac{\rho B^2 c}{D}Y_0\right] + \frac{iB^2c^2}{\omega_c D}Y_c \quad (4.26)$$

where

$$Y_e = \frac{(\mu^2 + \mu_5^2)K_e + 2\mu\mu_5 K_e^5}{T}$$
$$Y_0 = \frac{(\mu^2 + \mu_5^2)K_0 + 2\mu\mu_5 K_0^5}{T}$$
$$Y_c = \frac{(\mu^2 + \mu_5^2)K_c + 2\mu\mu_5 K_c^5}{T}$$

With this procedure we finally obtain three expressions that are well-behaved for the longitudinal conductivities of a two-currents chiral anomalous fluid. We can clearly see that, from this method, they all have the same structure of a sum of five terms: a constant, a non-anomalous term that depends on $1/\omega_m$ and three anomalous terms, each with a different relaxation parameter. To conclude our analysis we can quickly study some simplified results, to see under which conditions the various relaxation terms are needed.

- Limit I: $\rho_5 = \mu_5 = 0$ and consequently $s_1 = s_2 = 0$.

  This is the limit in which there is perfect chirality balance between the two nodes at equilibrium, so that the total chiral charge is zero and the left and right-handed node chemical potentials are at the same level. The two thermodynamics parameters s_1 and s_2 are set to zero for this limit to work. In this situation the three longitudinal conductivities

reduce to

$$\Sigma_{zz} = \sigma + \frac{i\rho^2}{\omega_m(\epsilon+p)} + \frac{iB^2c^2sT}{s_5(\epsilon+p)}\left(\frac{1}{\omega_c} - \frac{\mu\rho}{\omega_m(\epsilon+p)}\right)$$
$$\alpha_{zz} = -\frac{\mu\sigma}{T} + \frac{i\rho s}{\omega_m(\epsilon+p)} - \frac{iB^2c^2\mu s}{s_5(\epsilon+p)}\left(\frac{1}{\omega_c} - \frac{\mu\rho}{\omega_m(\epsilon+p)}\right)$$
$$\bar{\kappa}_{zz} = \frac{\mu^2\sigma}{T} + \frac{is^2T}{\omega_m(\epsilon+p)} + \frac{iB^2c^2\mu^2 s}{s_5(\epsilon+p)}\left(\frac{1}{\omega_c} - \frac{\mu\rho}{\omega_m(\epsilon+p)}\right)$$

and here we can see that, in this limit, all three conductivities still have an anomalous term that is bounded only when there is dissipation of charge and momentum (we argued in the previous section that the dissipation of energy was related to a non-zero axial chemical potential). Momentum dissipation, however, is needed to have a finite non-anomalous conductivity anyway.

- Limit II: $\rho = \mu = 0$ and, also, $f_5 = f_2 = 0$.

  This is the vacuum case, so that the net number of particles is zero. In this limit the conductivities reduce to

$$\Sigma_{zz} = \sigma + \frac{iB^2c^2}{e_5s_2 - e_2s_5}\left(\frac{s_2\mu_5}{\omega_e} - \frac{e_2}{\omega_c}\right)$$
$$\alpha_{zz} = -\frac{\mu_5\sigma_5}{T}$$
$$\bar{\kappa}_{zz} = \frac{\mu_5^2\sigma}{T} + \frac{is^2T}{\omega_m(\epsilon+p)} + \frac{iB^2c^2\mu_5^2}{T(e_5s_2 - e_2s_5)}\left(\frac{s_2\mu_5}{\omega_e} - \frac{e_2}{\omega_c}\right)$$

  We see that momentum dissipation is not needed for Σ_{zz} and α_{zz}, because it is always related to a non-zero charge density, while it is still relevant for the non-anomalous part of $\bar{\kappa}_{zz}$. In this situation the inter-valley scatterings still dissipate both energy and axial charge for the electrical and thermal conductivity, however the thermoelectric conductivity reduce to a constant term in this limit and it does not depend on the anomaly anymore.

- Limit III: $\rho = \rho_5 = \mu = \mu_5 = 0$ and also $f_5 = f_2 = s_1 = s_2 = 0$.

  This is the double zero density limit: the two nodes have no net charge and no net axial charge. In this state the conductivities become

$$\Sigma_{zz} = \sigma + \frac{iB^2c^2}{s_5\omega_c}$$
$$\alpha_{zz} = 0$$
$$\bar{\kappa}_{zz} = \frac{is^2T}{\omega_m(\epsilon+p)}$$

Here we can see that Σ_{zz} is the only conductivity that still has an anomalous term and that it is finite only if there is dissipation of axial charge. In this case the thermoelectric conductivity is zero, on the other hand the thermal conductivity is not zero, but it loses all the terms that depend on the anomaly. We can understand the anomalous term in Σ_{zz} noticing that, without dissipation, even when starting from a double zero equilibrium state, the anomaly keeps pumping more and more charge into the nodes. This mechanism keeps increasing the charge disparity and triggers the chiral magnetic effect that produces an infinite electric conductivity at long time-scale without axial charge dissipation.

Chapter 5

Final remarks and outlook

In our model we considered a Weyl semimetal as a two fluids system: one vector current and one axial current with a chiral anomaly. We studied the response of the system to small fluctuations near the thermodynamic equilibrium, studying the dynamics when under the action of a small electric field and temperature gradient. We also assumed that the linearized equations that describe the conservation of energy, momentum and axial charge are weakly broken by small parameters $1/\tau_i$, that are also needed to obtain finite longitudinal DC conductivities.

We gave a generalized definition for the heat current in the presence of multiple charges and, after a Laplace-Fourier transform, we were able to compute in linear response theory the full thermoelectric matrix from hydrodynamics. We managed to obtain the usual results found in literature [3, 46, 52] for the transverse conductivities in the presence of a weak magnetic field, up to the fact that we are working with two currents with different transport coefficients.

The results obtained for the zz components of Σ and $\bar{\alpha}$ (the electric conductivity and the thermoelectric conductivity that couples the electric field to the heat current) are also in perfect agreement with the results found in literature [3, 52], but the results obtained for α and $\bar{\kappa}$ still have some problems.

To find a well-behaved expression for $\bar{\kappa}_{zz}$ we took advantage of the Ward identities, that relate $\bar{\kappa}$ to Σ. With these final expressions we were able to discuss the form of the conductivities in some physically relevant simplified situations.

The computation we performed, that follows a very classic and well known

method, shows that it cannot easily account for dissipation of energy and charge when working at non-zero momentum, in fact the expressions obtained for α and $\bar{\kappa}$ do not reduce to the usual results found in literature. To correct this unexpected behaviour, that depends on the complicated algebra of the model, i.e. on how $\mathbf{k}$ couples to the dissipation constants, and obtain the usual magnetohydrodynamic results, we had to impose that τ_e and τ_c cannot enter any term that is not related to the chiral anomaly along the z axis.

Under these assumptions, however, we were able to correctly compute all the transverse transport coefficients when two currents are present, a result yet not find in literature to our knowledge.

Even if we followed a well known model with a well known procedure, the longitudinal results obtained for the terms related to the temperature gradients still have some clear issues.

First of all, as we mentioned, there is a constant term at $k_z = 0$ so that δJ_z and δQ_z are proportional to δT^0, showing that there is transport even without a temperature gradient. This term explicitly depends on the exact form of the anomalous coefficients, in fact we showed that this unexpected transport disappears when considering the transport coefficients as first order in the chemical potentials.

Secondly we observed that, even with all the dissipation constants turned on, the thermoelectric coefficients α_{zz} and $\bar{\kappa}_{zz}$ have a term that diverges as $1/\omega$ that again depends on the anomaly.

Lastly, there are some terms in these longitudinal expressions that are linear in ω and that thus disappear in the DC limit. Onsager relations require that $\bar{\alpha}_{zz} = \alpha_{zz}$ and this cannot be done if these terms are present.

All these problems arise because of the intricated couplings that the non-standard dissipation terms τ_e and τ_c have at non-zero momentum, that produce all these terms that would otherwise reduce to a very simple form or disappear completely.

The fix we found, to set $\tau_e = \tau_c$ to infinity in the non anomalous terms, might be hard to justify mathematically, however it seems to make sense physically and furthermore the other proposals found in literature have similar issues. All these problems might be a strong evidence that the procedure itself is not correct when multiple dissipations are present and when keeping $\mathbf{k} \neq \mathbf{0}$. It is also possible that the problem lies in the definition of the heat current: probably it is not straightforward to define the heat current when energy and charge are dissipated.

Furthermore, the Ward identities we used to find a well-behaved expression

for $\bar{\kappa}_{zz}$, naturally lead to a discrepancy between the result for $\bar{\kappa}_{zz}$ obtained from the linear response theory computation and the one from Ward identities. Because Ward identities are exact at all orders, independently on the specific constitutive expressions used, the two methods should produce the same results; the fact that this is not true is a strong evidence that there are some corrections to do, in particular extra care should be taken when defining the currents (specifically the heat current), the source terms and the constitutive relations.

It is possible that the expressions obtained from the Ward identities are just an approximation of the final results; in particular we could try to analyze the transport properties of a two-components chiral fluids from an holographic model to check the predictions that come from this approach and to better understand where the problem lies in the hydrodynamic computation.

It would also be interesting to follow instead the procedure used by Abbasi in [52], that is slightly different from the one used by [46] and from the one used in this thesis, to compute the thermoelectric coefficients of an anomalous fluid. In his work Abbasi used a different frame (the so called laboratory frame, that is closely related to the Landau frame and it is argued to be better suited for thermoelectric computations in the presence of a chiral anomaly, however this is likely not to have any major effect) for the constitutive relations, but more importantly he used a different approach to perturb the system with a temperature gradient. Instead of considering the fluctuation δT as the source for the temperature gradient, he explicitly defined a non-uniform temperature field $T(\mathbf{x})$ and defined the temperature gradient that is the source of the heat current as $\partial_z T(\mathbf{x})/T = \delta\zeta_z$, which is a first-order term in the fluctuations. This procedure seems to give better result when we also add dissipation of charge and energy to the system.

Another correction that could be interesting lies in the constitutive equations. The expressions we used for the stress-energy tensor and the currents in the Landau frame are the ones usually found in literature for a parity-invariant theory. However it is known that if parity is not a fundamental symmetry of the system, then we can in principle add other terms to the constitutive equations that depends on the Levi-Civita symbol [54]. Recently in [55] the authors showed that these Levi-Civita terms, called incoherent Hall conductivities, must be included in the constitutive equations to first order, even in a parity-invariant theory, if the magnetic field is present. It would be interesting to study the effect of these new terms to the computation we performed in this thesis.

Nonetheless, even with all the possible fixes, Onsager reciprocal relations are not automatically satisfied in our model. While the relations hold for the xx and xy components of the conductivities (and for the xy and yx components of the single matrices of course), Onsager relation is not automatically obeyed by the zz components.

This feature is in agreement with [52], who found the same problem with Onsager relations when dealing with a similar model, but with only one axial anomalous current $U(1)_A$. In that work the authors had an explicit expression for the equilibrium equation of state $p = p(T, \mu)$ in the kinetic regime, that allowed them to explicitly compute all the thermodynamic parameters of the system. From there they were able to find a set of constraints that must be obeyed by the dissipation time constants, in particular they found that for Onsager to be satisfied on the zz component all the dissipation rates had to be the same $\tau_e = \tau_m = \tau_c$.

On the other hand [4] used a different model in which the thermoelectric matrix was required to be Onsager reciprocal from the start, but even this model has its issues, since the results explicitly depend on the inverse of a matrix that is not invertible (even if the final expressions are well-defined).

Our model is very similar to the one used by [52] and it is possible that, given some more time to study the conditions needed to apply Onsager reciprocal relations, we could find the same constraints. Sadly the expressions for the zz components are very long and it is not easy to find a set of constraints that make sense with the time given. On the same line, it would also be interesting to see if we could find a very minimal expression for the equation of state of a two components chiral fluid, allowing us to explicitly calculate the thermoelectric parameters and thus reducing a lot the degrees of freedom of the system.

It would still be interesting to see if Onsager relations hold at least experimentally in a Weyl semimetal in the presence of an chiral anomaly. The idea behind the experiment is straightforward: we could take a Weyl semimetal with two Weyl cones of opposite chirality, like the one described by our model, and apply a constant magnetic field B in the direction that points from one node to the other (call it the z direction). By turning on a small electric field E along z we could be able to measure the longitudinal heat flux due to the thermoelectric effect and thus obtain $\bar{\alpha}_{zz}$. If instead a temperature gradient along the z direction is created, we could measure the electric current produced by the Seeback effect and obtain α_{zz} (see 5.1).

If the result of the experiment were to confirm that in a Weyl semimet-

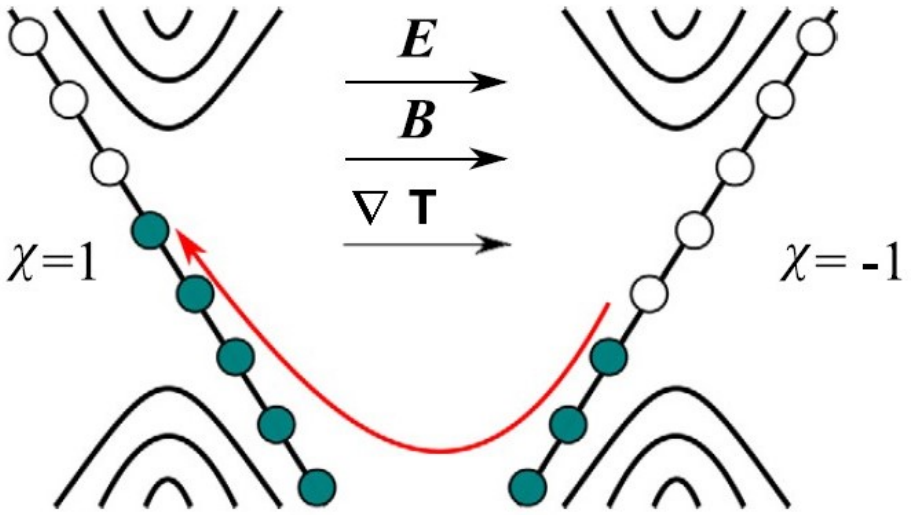

Figure 5.1: The basic structure of the experiment. By turning on an electric field and then a temperature gradient parallel to the magnetic field we can measure the two thermoelectric longitudinal conductivities α_{zz} and $\bar{\alpha}_{zz}$ for the Weyl semimetal, so that we can verify experimentally if Onsager reciprocal relations are satisfied.

als Onsager reciprocal relations still hold, then we must look for a set of constraints to fix the thermodynamics of our model. Some thermodynamic constraints are probably very likely to exists, since as we mentioned there are a lot of free equilibrium parameters in our system at the moment, but we expect an equilibrium state to have fewer degrees of freedom, but other constraints, like the ones obtained by [52], $\tau_e = \tau_m = \tau_c$ are really hard to justify physically.

The three time constants are related to different microscopical phenomena and it is very unclear why they should be the same. Furthermore, as we see in our work, τ_e and τ_c seems to depend on the anomaly alone, while τ_m is always present. In kinetic approaches usually τ_e and τ_c are considered as depending on the inter-valley scattering between the two Weyl nodes, while τ_m comes from the intra-valley scattering of the particles of same chirality in each node (or, like [46], simply requiring that there are some impurities in the crystal so that the background potential is not perfectly periodic).

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