A new class of hydrostatic theories

based on 2211.XXXXX with A. Amoretti, D.K. Brattan and I. Matthaiakakis

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Luca Martinoia

University of Genova

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My research

Hydrodynamics:

- Hydro with (pseudo-)Goldstone for broken translations
- Hydro with ABJ anomaly
- *A new class of hydrostatic theories*

Non hydrodynamics:

- Electric field and thin films superconductors
- (torsional anomaly)

Hydrodynamics

- Many-body EFT for systems in LTE (long wavelength/time scale)
- Used for: QGP, Neutron star mergers, condensed matter, ...

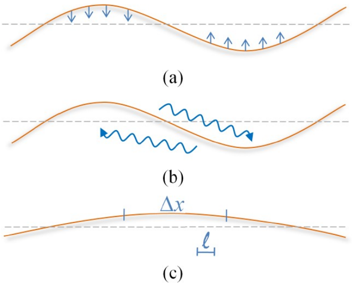

Figure: (a) Non conserved quantities relax quickly. (b) Conserved quantities can only reach equilibrium via transport. (c) Local thermodynamic equilibrium, $l_{ee} \ll l_{e\gamma}, l_{e,\mathrm{imp}}$.

Hydrodynamics

- Degrees of freedom: conserved charges

$$\partial_\mu T^{\mu\nu} = F^{\nu\lambda} J_\lambda \qquad \partial_\mu J^\mu = 0$$

- Constitutive relations: currents in terms of $T(t,\vec{x}), \mu(t,\vec{x}), u^\mu(t,\vec{x})$ based on
  - symmetries
  - derivative expansion $T^{\mu\nu}, J^\mu \sim \mathcal{O}(\partial^0) + \mathcal{O}(\partial) + \ldots$
  - second law of thermodynamics $\partial_\mu S^\mu \geq 0 \quad \Rightarrow \quad \sigma_0, \eta, \zeta \geq 0$

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + p\Delta^{\mu\nu} - \eta \Delta^{\mu\alpha}\Delta^{\nu\beta}\sigma_{\alpha\beta} - \zeta \Delta^{\mu\nu}\partial_\alpha u^\alpha + \mathcal{O}(\partial^2)$$

$$J^\mu = n u^\mu + \sigma_0 \Delta^{\mu\nu}\left(E_\nu - T\partial_\nu \frac{\mu}{T}\right) + \mathcal{O}(\partial^2)$$

Conductivity and momentum relaxation

From linear response theory and hydrodynamics the conductivity $\vec{J} = \sigma\vec{\mathbb{E}}$ is divergent as $\omega \to 0$

$$\sigma(\omega) = \sigma_0 + \frac{in^2}{(p+\epsilon)\omega}$$

because of translation invariance.

Adding momentum relaxation by hand [Hartnoll *et al.* 2007]

$$\partial_\mu T^{\mu i} = F^{i\nu} J_\nu + \Gamma T^{0i}$$

the conductivity is now finite in DC and Drude-like

$$\sigma(\omega) = \sigma_0 + \frac{n^2}{(p+\epsilon)(\Gamma - i\omega)} \qquad \to \qquad \sigma_{DC} = \sigma_0 + \frac{n^2}{\Gamma(p+\epsilon)}$$

Motivation

Theory:

- Γ breaks boosts (and translations), yet the constitutive relations have Lorentz symmetry
- Hydrodynamics predicts stationary states with $E_\nu - T\partial_\nu \frac{\mu}{T} = 0$ and, with momentum relaxation, $\vec{v} = 0$
- $\Gamma \sim \mathcal{O}(\partial)$, but then should disappear on stationary states

Phenomenology:

- DC measurements consider open systems that exchange heat with environment and reach a steady state
- Paradigm: Drude model
- Polarization

Motivation

Our idea:

- Focus on hydrostatic solutions
- System has energy and momentum sinks to compensate the effects of E_μ and reaches hydrostatic state
- States depend on velocity: boosts are broken by Γ
- Weak sinks: relevant DoF are not modified
- Derivative counting: E_μ and Γ order zero $\mathcal{O}(\partial^0)$

The hydrodstatic generating functional [Jensen *et al.* 2012]

- Hydrostatic (non dissipative): $\exists$ time-like Killing vector β^μ such that $\mathcal{L}_\beta A_\mu = \mathcal{L}_\beta g_{\mu\nu} = 0, \ldots$
- Diffeomorphism and gauge invariant order zero fields

$$T = \frac{T_0}{\sqrt{-\beta^2}} \qquad \mu = T(\beta^\mu A_\mu + \Lambda) \qquad u^\mu = \frac{\beta^\mu}{\sqrt{-\beta^2}}$$

- Hydrostatic conditions, e.g. $\partial_\mu u^\mu = 0$ and $E_\nu - \partial_\nu \mu = \mu a_\nu$

Then

$$W = \int \mathrm{d}^d x \, \sqrt{-g} \left[p(s_0) + \sum_{n=1}^{m} \sum_{i=1}^{N_m} \alpha_{n,i}(s_0) s_{n,i} \right]$$

and $T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta W}{\delta g_{\mu\nu}}$, $J^\mu = \frac{1}{\sqrt{-g}} \frac{\delta W}{\delta A_\mu}$

The plan

- Use generating functional to obtain constitutive relations (at order zero in derivatives) without relaxations
- Add by hand relaxation terms Γ
- Check how hydrostatic constraint are modified and check for self-consistency
    - Are the various terms independent?
    - Are there bounds on entropy production?
    - Hydrodynamic modes? Stability?
- Repeat for order one (no dissipation yet)
    - Do the hydrostatic constraints receive derivative corrections?

Boost-agnostic thermodynamics [de Boer *et al.* 2018]

If thermodynamics depends on the velocity we have

$$p = p(T, \mu, \vec{\mathbb{E}}^2, \vec{v}^2, \vec{v} \cdot \vec{\mathbb{E}})$$

we define

$$s = \left.\frac{\partial p}{\partial T}\right|_{\mu, \vec{\mathbb{E}}, \vec{v}} \qquad n = \left.\frac{\partial p}{\partial \mu}\right|_{T, \vec{\mathbb{E}}, \vec{v}}$$

$$\vec{\mathbb{P}} = \left.\frac{\partial p}{\partial \vec{\mathbb{E}}}\right|_{\mu, T, \vec{v}} = \kappa_{\mathbb{E}} \vec{\mathbb{E}} + \beta_{\mathbb{P}} \vec{v} \qquad \vec{P} = \left.\frac{\partial p}{\partial \vec{v}}\right|_{\mu, \vec{\mathbb{E}}, T} = \beta_{\mathbb{P}} \vec{\mathbb{E}} + \rho_m \vec{v}$$

so that Euler and Gibbs-Duhem relations are

$$\mathrm{d}p = s\,\mathrm{d}T + n\,\mathrm{d}\mu + P_i\,\mathrm{d}v^i + \mathbb{P}^i\,\mathrm{d}\mathbb{E}_i$$

$$\epsilon + p = sT + n\mu + \vec{\mathbb{E}} \cdot \vec{\mathbb{P}} + \vec{v} \cdot \vec{P}$$

Aristotelian geometry [de Boer *et al.* 2018, Penrose 1968]

Clock 1-form τ_μ and spatial metric $h_{\mu\nu}$ with signature $(0, 1, \ldots, 1)$.

We can decompose $h_{\mu\nu} = \delta_{ab} e^a_\mu e^b_\nu$ and invert the matrix (τ_μ, e^a_μ) to get $(-\nu^\mu, e^\mu_a)$

$$\nu^\mu \tau_\mu = -1 \qquad \nu^\mu e^a_\mu = 0 \qquad e^\mu_a \tau_\mu = 0 \qquad e^\mu_a e^b_\mu = \delta^a_b$$

Completeness relation

$$-\nu^\mu \tau_\nu + e^\mu_a e^a_\nu = \delta^\mu_\nu$$

define $h^{\mu\nu} = \delta^{ab} e^\mu_a e^\nu_b$.

We can define metric-compatible covariant derivatives, but with non-zero torsion.

The order zero – a minimal model

At order zero in derivatives (without Γ)

$$T^0{}_0 = -\epsilon \qquad T^0{}_i = P_i$$
$$T^i{}_0 = -(\epsilon + p - \vec{\mathbb{P}} \cdot \vec{\mathbb{E}})v^i \qquad T^i{}_j = p\delta^i_j + \rho_m v^i v_j - \kappa_{\mathbb{E}} \mathbb{E}^i \mathbb{E}_j$$
$$J^0 = n - \partial_j \mathbb{P}^j \qquad J^i = nv^i + \partial_t \mathbb{P}^i$$

Now add energy and momentum relaxations

$$\partial_\mu T^\mu{}_0 = F_{0\nu} J^\nu + \Gamma_\epsilon T^0{}_0 \qquad \partial_\mu T^\mu{}_i = F_{i\nu} J^\nu + \Gamma T^0{}_i$$

From the EoM at order zero we postulate

$$n(\mathbb{E}^i - \partial^i \mu) = \Gamma P^i$$

and for consistency

$$\Gamma_\epsilon \epsilon = \Gamma \vec{P} \cdot \vec{v}$$

The order zero – other terms

Hydrodynamics as EFT: at order zero we should consider a momentum-polarization relaxation $\Gamma_{\mathbb{P}}\mathbb{P}^i$.
Order zero EoM reads

$$
\begin{aligned}
n v^i (\mathbb{E}_i - \partial_i \mu) &= \Gamma_\epsilon \epsilon + \mathcal{O}(\partial) \\
n (\mathbb{E}_i - \partial_i \mu) &= \Gamma_{\mathbb{P}} \mathbb{P}_i + \Gamma P_i + \mathcal{O}(\partial)
\end{aligned}
$$

so that

$$
\begin{aligned}
\vec{v} &= \left(\frac{n - \kappa_{\mathbb{E}} \Gamma_{\mathbb{P}} - \beta_{\mathbb{P}} \Gamma}{\beta_{\mathbb{P}} \Gamma_{\mathbb{P}} + \rho_m \Gamma} \right) \vec{E} - \frac{n}{\beta_{\mathbb{P}} \Gamma_{\mathbb{P}} + \rho_m \Gamma} \vec{\partial} \mu \\
\Gamma_\epsilon \epsilon &= \vec{v} \cdot \left(\Gamma \vec{P} + \Gamma_{\mathbb{P}} \vec{\mathbb{P}} \right)
\end{aligned}
$$

Entropy, modes and order one

- Entropy is conserved $\partial_\mu S^\mu = 0$
- At order one there are corrections, but $n(\mathbb{E}_i - \partial_i \mu) = \Gamma_{\mathbb{P}} \mathbb{P}_i + \Gamma P_i$ still holds
- Modes: two gapless and non-propagating, one gapped and propagating (ω_1), one gapped and non-propagating (ω_2)

$$\omega_1 = -i\Gamma_{\text{eff}} + \left(\frac{(n - \beta_{\mathbb{P}}\Gamma_P - \kappa_{\mathbb{E}}\Gamma_{\mathbb{P}})v}{(\partial_T \epsilon \partial_\mu n - \partial_\mu \epsilon \partial_T n)\, \rho_m^2 \Gamma_{\text{eff}}} \right) \vec{\mathbb{E}} \cdot \vec{k}$$

$$\omega_2 = -i\Gamma_{\text{eff}}$$

$$\Gamma_{\text{eff}} = \frac{1}{\rho_m} \left(\beta_{\mathbb{P}} \Gamma_{\mathbb{P}} + \rho_m \Gamma \right)$$

- Stability (causality): $\beta_{\mathbb{P}} \Gamma_{\mathbb{P}} + \rho_m \Gamma \geq 0$

Future plans

In standard hydrodynamics there are dissipative terms $\mathcal{O}(\partial)$

$$J_i \sim \sigma_0 \left(\mathbb{E}_i - \partial_i \mu\right)$$

From our new hydrostatic class: $n\mathbb{E}_i - n\partial_i\mu - \Gamma_{\mathbb{P}}\mathbb{P}_i - \Gamma P_i \sim \mathcal{O}(\partial)$, so we should have

$$J_i \sim \sigma_0 \left(\mathbb{E}_i - \partial_i \mu - \Gamma_{\mathbb{P}}\mathbb{P}_i - \Gamma P_i\right)$$

$\Longrightarrow$ New expressions for the thermoelectric conductivities

Summary

- Standard hydrodynamics is not suitable to describe systems with relaxations
- Hydrostatic generating functional also fails
- We systematically studied all possible relaxation terms order by order in hydrostatic regime
- Stationary conditions must be modified $n(\mathbb{E}_i - \partial_i \mu) = \Gamma_{\mathbb{P}} \mathbb{P}_i + \Gamma P_i$
- We predict background DC conductivities on these stationary states $\vec{J} \sim \vec{v} \sim \vec{E}$
- New expressions for the optical conductivities

Thanks for the attention!

Hydrostatic conditions

All scalars obey $\mathcal{L}_\beta(\dots) = T\mathcal{L}_u(\dots) = (\partial_t + v^i\partial_i)(\dots) = 0$.
All tensors obey $\mathcal{L}_\beta(A_\mu, \tau_\mu, h_{\mu\nu}, \dots) = 0$.

From their definitions and Bianchi identity one finds (flat spacetime)

$$\partial_\mu T = 0 \qquad \partial_t v^i = 0 \qquad \partial_i v_j + \partial_j v_i = 0$$
$$\partial_i \mathbb{E}_j - \partial_j \mathbb{E}_i = 0 \qquad \mathbb{E}_i - \partial_i \mu = 0$$

Boost-agnostic hydrostatic generating functional

Generating functional: β^μ time-like Killing vector $\mathcal{L}_\beta(\dots) = 0$
Geometrise thermodynamics

$$T = \frac{1}{\tau_\mu \beta^\mu} \qquad \mu = T(\beta^\mu A_\mu + \Lambda) \qquad u^\mu = T\beta^\mu$$

Then

$$W[\tau, h, A, F] = \int \mathrm{d}^d x \; ep(T, \mu, \mathbb{E}^2, u^2, \mathbb{E}\cdot u)$$
$$\delta W = \int \mathrm{d}^d x \; e\left(-T^\mu \delta\tau_\mu + \frac{1}{2} T^{\mu\nu}\delta h_{\mu\nu} + J^\mu \delta A_\mu + \frac{1}{2} M^{\mu\nu}\delta F_{\mu\nu}\right)$$

where $T^\mu_\nu = -T^\mu \tau_\nu + T^{\mu\rho} h_{\rho\nu}$ and $M^{\mu\nu} = \nu^\mu \mathbb{P}^\nu - \nu^\mu \mathbb{P}^\mu$.

Covariant derivative

Metric compatibility

$$\nabla_\mu \tau_\nu = 0 \qquad \nabla_\mu h^{\nu\rho} = 0$$
$$\nabla_\mu \nu^\nu = 0 \qquad \nabla_\mu h_{\nu\rho} = 0$$

gives

$$\Gamma^\lambda_{\mu\nu} = -\nu^\lambda \partial_\mu \tau_\nu + \frac{1}{2} h^{\lambda\kappa} \left(\partial_\mu h_{\nu\kappa} + \partial_\nu h_{\mu\kappa} - \partial_\kappa h_{\mu\nu} \right) - h^{\lambda\kappa} \tau_\nu K_{\mu\kappa}$$

with $K_{\mu\nu} = -\frac{1}{2}\mathcal{L}_\nu h_{\mu\nu}$ the extrinsic curvature. Has non-zero torsion.

Flat space limit

$$\tau_\mu = \delta^0_\mu \qquad h_{\mu\nu} = \delta^i_\mu \delta^i_\nu \qquad \nu^\mu = -\delta^\mu_0 \qquad h^{\mu\nu} = \delta^\mu_i \delta^\nu_i$$

First order corrections

First order there are 14 independent scalars

$$s_{(1)} = \Big\{ \nu^\mu \partial_\mu \left(T, \mu, u^2, \mathbb{E}^2, u \cdot \mathbb{E} \right), \mathbb{E}^\mu \partial_\mu \left(T, \mu, u^2, \mathbb{E}^2, u \cdot \mathbb{E} \right),$$
$$u^\mu u^\nu \nabla_\mu \mathbb{E}_\nu, \mathbb{E}^\mu u^\nu \nabla_\mu \mathbb{E}_\nu, \nabla_\mu \mathbb{E}^\mu, \partial_\mu \tau_\nu \nu^{[\mu} h^{\nu]\sigma} \mathbb{E}_\sigma,$$
$$\partial_\mu (h_{\nu\sigma} u^\sigma) \nu^{[\mu} h^{\nu]\rho} \mathbb{E}_\rho \Big\}$$

Then

$$W_{(1)} = \int \mathrm{d}^d x \; e \sum_i F_i(T, \mu, \mathbb{E}^2, u^2, \mathbb{E} \cdot u) s_{(1)}^{(i)}$$